Chapter 0

Multiple integration methods

1. Integration of Improper fractions

A term is called an improper fraction when the degree of the numerator is greater or equal to the one of the denominator. To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction.

$$\frac{x^4+3x^2}{x^2+2x+1} = x^2-2x+3-\frac{10x+6}{x^2+2x+1} = x^2-2x+3+\frac{4}{(1+x)^2}-\frac{10}{1+x}$$

So we'll learn to integrate fractions (the remainder) that's denominator degree is greater than the numerator ones

1er Case: The denominators factors are of first order and distinct

For each of the unrepeated factors such as *x - a* corresponds a simple fraction of this form:

$$\frac{A}{x-a}$$

Where A is a constant.

EXAMPLE $\int \frac{2x+3}{x^3+x^2-2x}dx$

$$x^3+x^2-2x = x(x-1)(x+2)$$

$$\frac{2x+3}{x^3+x^2-2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{(A+B+C)x^2+(A+2B-C)x-2A}{x(x-1)(x+2)}$$

By identification:

$$\frac{2x+3}{x^3+x^2-2x} = -\frac{3}{2x} + \frac{5}{3(x-1)} - \frac{1}{6(x+2)}$$

$$\int \frac{2x+3}{x^3+x^2-2x}dx = -\frac{3}{2}\ln|x| + \frac{5}{3}\ln|x-1| - \frac{1}{6}\ln|x+2| + c$$

NB:

A quick method to get the constants A, B et C consists on cancelling the factors in the following equations:

$$2x+3 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

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$$x = -2 \Rightarrow -1 = -2\times -3\times C \Rightarrow C = -\frac{1}{6}$$

2nd Case: The denominators factors are of first order and some of them are repeated

For each of the repeated factors such as $(x - a)^n$ corresponds n simple fractions:

$$\frac{A_n}{(x-a)^n}+\frac{A_{n-1}}{(x-a)^{n-1}}+\ldots+\frac{A_1}{(x-a)}$$

EXAMPLE

$$\frac{x^3+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{(x-1)^3}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)}=-\frac{1}{x}+\frac{2}{(x-1)^3}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)}$$

3rd Case: The denominator contains distinct 2nd order factors

For each unrepeated quadratic factor x^2+px+q corresponds a simple fraction of this form:

$$\frac{Ax+B}{x^2+px+q}$$

We integrate this form by completing the square in the denominator:

$$x^2+px+q=\left(x+\frac{p}{2}\right)^2+\frac{1}{4}\left(4q-p^2\right)$$

Then by substituting $u=x+\frac{p}{2}$

EXAMPLE

$$\frac{4}{x\left(x^2+4\right)}=\frac{A}{x}+\frac{Bx+c}{x^2+4}=\frac{1}{x}-\frac{x}{x^2+4}$$

$$\int\frac{4}{x\left(x^2+4\right)}dx=\int\frac{1}{x}dx-\int\frac{x}{x^2+4}dx=\ln|x|-\frac{1}{2}\int\frac{du}{u}=\ln|x|-\frac{1}{2}\ln\left(x^2+4\right)+c$$

4th Case: The denominator contains 2nd order factors and some of them are repeated

For each of the repeated quadratic factors such as $(x^2 + px + q)^n$ corresponds n simple fractions:

$$\frac{A_n x + B_n}{\left(x^2 + px + q\right)^n} + \frac{A_{n-1} x + B_{n-1}}{\left(x^2 + px + q\right)^{n-1}} + \ldots + \frac{A_1 x + B_1}{x^2 + px + q}$$

EXAMPLE

$$\frac{2x^3 + x + 3}{\left(x^2+1\right)^2} = \frac{Ax+B}{\left(x^2+1\right)^2} + \frac{Cx+D}{\left(x^2+1\right)} = \frac{3}{\left(x^2+1\right)^2} - \frac{x}{\left(x^2+1\right)^2} + \frac{2x}{\left(x^2+1\right)}$$

$$J = \int \frac{2x^3 + x + 3}{\left(x^2+1\right)^2} dx = \int \left(\frac{3}{\left(x^2+1\right)^2} - \frac{x}{\left(x^2+1\right)^2} + \frac{2x}{\left(x^2+1\right)} \right) dx$$

$$J = \int \frac{3}{\left(x^2+1\right)^2} dx + \frac{1}{2}\frac{1}{1+x^2} + Log\left(1+x^2\right)$$

$$\frac{3}{\left(x^2+1\right)^2} = 3\frac{1}{\left(x^2+1\right)^2} = 3\frac{1+x^2-x^2}{\left(x^2+1\right)^2} = 3\left(\frac{1}{\left(x^2+1\right)} - \frac{x^2}{\left(x^2+1\right)^2}\right)$$

$$\int \frac{3}{\left(x^2+1\right)^2} dx = 3\int \left(\frac{1}{x^2+1} - \frac{x^2}{\left(x^2+1\right)^2} \right) dx = 3ArcTanx - 3\int \frac{x^2}{\left(x^2+1\right)^2} dx$$

$$J = 3ArcTanx + \frac{1}{2}\frac{1}{1+x^2} + Log\left(1+x^2\right) - 3\int \frac{x^2}{\left(x^2+1\right)^2} dx$$

$$u = x \Rightarrow du = dx$$

$$dv = \frac{xdx}{\left(x^2+1\right)^2} \Rightarrow v = -\frac{1}{2}\frac{1}{x^2+1}$$

$$\int \frac{x^2}{\left(x^2+1\right)^2} dx = -\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\int \frac{1}{x^2+1} dx = -\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2} ArcTanx$$

$$J = 3ArcTanx + \frac{1}{2}\frac{1}{1+x^2} + Log\left(1+x^2\right) - 3\left(-\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2} ArcTanx\right)$$

$$J = \frac{3}{2} ArcTanx + Log\left(1+x^2\right) + \frac{3x - +1}{2\left(x^2+1\right)}$$

2. Integration by variable substitution

Differential only containing fractional exponents of x

We reduce this expression by using the following substitution $x = t^n$, where n is the lower common denominator of x fractional exponents.

EXAMPLE

$$\frac{\sqrt{x}}{1+\sqrt[4]{x^3}} = \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{4}}}.$$

Let's use $x = t^4 \Rightarrow$

$$\int \frac{\sqrt{x}}{1+\sqrt[4]{x^3}} dx = \int \frac{t^2}{1+t^3} 4t^3 dt = \frac{4}{3} x^{\frac{3}{4}} - \frac{4}{3} Log\left(1+x^{\frac{3}{4}}\right) + c$$

Differential only containing fractional exponents of a+bx

We reduce this expression by using the following substitution $a + bx = t^n$, where n is the lower common denominator of a+bx fractional exponents.

EXAMPLE

$$\int \frac{dx}{(1+x)^{3/2} + (1+x)^{1/2}}$$

Let's use $1 + x = t^2$

$$\int \frac{dx}{(1+x)^{3/2} + (1+x)^{1/2}} = \int \frac{2dt}{t^2+1} = 2ArcTan(t) + c2ArcTan\left((1+x)^{1/2}\right) + c$$

3. Binomial differential

A binomial differential is of the following form:

$$x^m\left(a+bx^n\right)^p dx$$

We can reduce this differential to the following form

$$x^m\left(a+bx^n\right)^{\frac{r}{s}} dx$$

where m, n, r, s are integers and n is positive.

In fact, if m and n are fractions, let's choose α in such a way that mα and nα be integers. (α lower common multiple of m and n denominators).

Let's consider x=t$^{\alpha}$. $\Rightarrow x^m\left(a+bx^n\right)^{\frac{r}{s}} dx = at^{m\alpha+\alpha-1}\left(a+bt^{n\alpha}\right)^p$

x exponents are consequently replaced by integers. In the same way:

$$x^m\left(a+bx^n\right)^p dx = x^{m+np}\left(ax^{-n}+b\right)^p$$

Shows that whatever the sign of n is, the exponent of x can be turned positive.

"Binôme" differential of the form

$$x^m\left(a+bx^n\right)^{\frac{r}{s}} dx$$

In which m,n,r,s are integers and n is positive.

1st Case : $\dfrac{m+1}{n} = entier \Rightarrow$ we put $a+bx^n = t^s$

EXAMPLE

$$\int \frac{x^3}{\left(a+bx^2\right)^{3/2}} dx$$

Here m=3, n=2, r =3 and s=2. $\dfrac{m+1}{n}=\dfrac{4}{2}=2=\text{integer}$.

We put $a+bx^2 = t^2 \Rightarrow x = \left(\dfrac{t^2-a}{b}\right)^{1/2} \Rightarrow dx = \dfrac{tdt}{\sqrt{b\left(t^2-a\right)}}$

$$\int \frac{x^3}{\left(a+bx^2\right)^{3/2}}dx = \int \left(\frac{t^2-a}{b}\right)^{3/2} \times \frac{1}{t^3} \times \frac{tdt}{b^{1/2}\left(t^2-a\right)^{1/2}} = \frac{1}{b^2} \times \int \frac{\left(t^2-a\right)}{t^2}dt = \frac{1}{b^2}\left(t+\frac{a}{t}\right)+c$$

$$\int \frac{x^3}{\left(a+bx^2\right)^{3/2}}dx = \frac{1}{b^2}\left(\sqrt{a+bx^2}+\frac{a}{\sqrt{a+bx^2}}\right)+c$$

2nd Case : $\frac{m+1}{n}+\frac{r}{s} = entier\ ou\ 0 \Rightarrow$ we use $a+bx^n = t^s x^n$

EXAMPLE

$$\int \frac{dx}{x^4\sqrt{1+x^2}}$$

here m=-4, n=2, r=-1, s=2 $\Rightarrow$ we use

$$1+x^2 = t^2x^2 \Rightarrow x^2 = \frac{1}{t^2-1} \text{ et } t = \left(\frac{1+x^2}{x^2}\right)^{1/2}$$

$$\Rightarrow dx = -\frac{tdt}{\left(t^2-1\right)^{3/2}} \Rightarrow \int \frac{dx}{x^4\sqrt{1+x^2}} = -\int \left(t^2-1\right)dt = t-\frac{t^3}{3}+c = \frac{\left(2x^2-1\right)\left(1+x^2\right)^{1/2}}{2x^3}+c$$

4. Trigonometric integrals

A rational trigonometric differential containing sin(u) et cos(u) can be transformed in a simpler differential expression using the following substitution :

$$Tan\left(\frac{u}{2}\right) = t$$

or one of these substitutions :

$$\sin(u) = \frac{2t}{1+t^2}; \quad \cos(u) = \frac{1-t^2}{1+t^2}; \quad \mathrm{Tan}(u) = \frac{2t}{1-t^2}; \quad du = \frac{2dt}{1+t^2}$$

EXAMPLE

$$\int \frac{dx}{5+4\sin 2x}$$

Let's use $\sin(2x)=\frac{2t}{1+t^2} \Rightarrow \quad du=\frac{2dt}{1+t^2}$

$$\int \frac{dx}{5+4\sin 2x}=\int \frac{dt}{5t^2+8t+5}=\frac{1}{3}ArcTan\left(\frac{5t+4}{3}\right)+c=\frac{1}{3}ArcTan(u)+c$$

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5. Integration of expressions containing $\sqrt{a^2-u^2}$ or $\sqrt{u^2 \pm a^2}$

When we have $\sqrt{a^2-u^2}$ **we use**		$u=a\sin(z)$
When we have $\sqrt{u^2+a^2}$ **we use**		$u=aTan(z)$
When we have $\sqrt{u^2-a^2}$ **we use**		$u=\frac{a}{\cos^2(z)}$

EXAMPLE 1

METHODE 1

- $I=\int \sqrt{1+x^2}\,dx$

  Let's use $u=\sqrt{1+x^2}$ and $dv=dx$

  $$I=\int \sqrt{1+x^2}\,dx=x\sqrt{1+x^2}-\int \frac{x^2}{\sqrt{1+x^2}}dx$$

  $$\int \frac{x^2}{\sqrt{1+x^2}}dx=\int \frac{x^2+1-1}{\sqrt{1+x^2}}dx=\int \sqrt{1+x^2}\,dx-\int \frac{1}{\sqrt{1+x^2}}dx=I-ArcSinh(x)$$

  $$I=x\sqrt{1+x^2}-I+ArcSinh(x)$$

  $$2I=x\sqrt{1+x^2}+ArcSinh(x)$$

  $$I=\frac{1}{2}\left(x\sqrt{1+x^2}+ArcSinh(x)\right)$$

METHODE 2

- $I=\int \sqrt{1+u^2}\,du$

  Let's use $u=Tan(x)$

$$I = \int \frac{1}{\cos^3 x} dx = \int \frac{\cos^2 x + \sin^2 x}{\cos^3 x} dx = \int \left(\frac{1}{\cos x} + \frac{\sin^2 x}{\cos^3 x} \right) dx = K + J$$

$$K = \int \frac{1}{\cos x} dx = \int \frac{\frac{2}{1+t^2}}{\frac{1-t^2}{1+t^2}} dt = \int \frac{2}{1-t^2} dt = \int \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt = Log\left(1+t\right) - Log\left(1-t\right)$$

$$K = Log\left(\frac{1+t}{1-t}\right)$$

$$J = \int \frac{\sin^2 x}{\cos^3 x} dx = \int \frac{\left(\frac{2t}{1+t^2}\right)^2}{\left(\frac{1-t^2}{1+t^2}\right)^3} \frac{2}{1+t^2} dt = \int \frac{8t^2}{\left(1-t^2\right)^3} dt$$

$$\frac{8t^2}{\left(1-t^2\right)^3} = \frac{1}{\left(t+1\right)^3} - \frac{1}{\left(t-1\right)^3} - \frac{1}{2\left(t-1\right)^2} - \frac{1}{2\left(t+1\right)^2} + \frac{1}{2\left(t-1\right)} - \frac{1}{2\left(t+1\right)}$$

$$J = \int \frac{8t^2}{\left(1-t^2\right)^3} dt = \frac{1}{2}\left(\frac{2\left(t+t^3\right)}{\left(t^2-1\right)^2} + Log\left(t-1\right) - Log\left(t+1\right) \right)$$

$$J = \frac{1}{2}\left(\frac{2\left(t+t^3\right)}{\left(t^2-1\right)^2} + Log\left(\frac{t-1}{t+1}\right) \right)$$

$$K + J = \frac{1}{2}\left(\frac{2\left(t+t^3\right)}{\left(t^2-1\right)^2} \right) + \frac{1}{2} Log\left(\frac{t-1}{t+1}\right) + \frac{1}{2} Log\left(\frac{1+t}{1-t}\right) + \frac{1}{2} Log\left(\frac{1+t}{1-t}\right)$$

$$I = \frac{1}{2}\left(\frac{2\left(t+t^3\right)}{\left(t^2-1\right)^2} \right) + \frac{1}{2} Log\left(\frac{1+t}{1-t}\right)$$

EXAMPLE 2

- $\int \frac{dx}{\left(a^2 - x^2\right)^{3/2}}$

Let's use $x = a\sin(z)$

$$\int \frac{dx}{\left(\sqrt{a^2 - x^2}\right)^3} = \int \frac{a\cos(z)dz}{a^3\cos^3(z)} = \frac{1}{a^2}\int \frac{dz}{\cos^2(z)} = \frac{Tan(z)}{a^2} + c = \frac{1}{a^2}\frac{\sin(z)}{\cos(z)} + c$$

$$\int \frac{dx}{\left(\sqrt{a^2-x^2}\right)^3} = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + c$$

EXAMPLE 3

- $\int \frac{dx}{x\sqrt{4x^2+9}}$

Let's use $u = 2x \Rightarrow$

$$\int \frac{dx}{x\sqrt{4x^2+9}} = \int \frac{du}{u\sqrt{u^2+9}}$$

Let's use $u = 3Tan(z)$

$$\Rightarrow \int \frac{du}{u\sqrt{u^2+9}} = \int \frac{3dz}{\cos^2(z)\times 3Tan(z)\times\sqrt{9Tan^2(z)+9}} = \frac{1}{3}\int \frac{dz}{\sin(z)}$$

$$\frac{1}{3}\int \frac{dz}{\sin(z)} = \frac{1}{3}\int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}}dt = \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3}Log(t) + c = \frac{1}{3}Log\left(\frac{-3+\sqrt{9+u^2}}{u}\right) + c$$

$$\int \frac{dx}{x\sqrt{4x^2+9}} = \frac{1}{3}Log\left(\frac{-3+\sqrt{9+4x^2}}{2x}\right) + c$$