Chapter X

Generalized or improper integrals

1. Definition of the generalized integral

- The integral $\int_a^b f(x)dx$ is said generalized or improper of first species, if:

  $a = -\infty$ or $b = +\infty$ or both

- The integral $\int_a^b f(x)dx$ is said generalized or improper of second species, if:

  $f(x)$ is not bounded at a or many points of the interval [a, b] ; such points are called singularities of f(x).

- The integral $\int_a^b f(x)dx$ is said generalized or improper of third species, if it's at the same time of first and second species.

EXAMPLES

1) $\int_0^{+\infty} x^2 dx$ 2) $\int_0^4 \frac{dx}{x-4}$ 3) $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{x}} dx$

2. Generalized integrals of first species

Definition 1

Land f(x) a bounded and integrable function on every finite interval [a, b]. So by definition:

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx$$

If this limit exists, the integral is said convergent; otherwise the integral diverges, that's the end of the story.

Definition 2

Similarly, $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$

The integral is said convergent if the limit exists.

EXAMPLE $I = \int_1^{+\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^{+\infty} = \lim_{x\to+\infty}\left(-\frac{1}{x}\right) + 1 = 1 \Rightarrow$ I is convergent.

Definition 3

Similarly we define:

$$I=\int_{-\infty}^{+\infty} f(x)dx=\int_{-\infty}^{t} f(x)dx+\int_{t}^{+\infty} f(x)dx \quad \text{where t is a real number}$$

If both integrals are convergent then I is convergent.

If one diverges that I is divergent.

If both are divergent but their sum converges then I converge.

2.1. Generalized integral of particular functions

2.1.1. Geometrical or exponential integral

The integral $\int_{0}^{+\infty} e^{-\alpha x}dx$ where α is a constant is said geometrical integral. We can easily prove that $\int_{0}^{+\infty} e^{-\alpha x}dx=\begin{cases}\text{convergent} & \text{if} \quad \alpha>0\\ \text{divergent} & \text{if} \quad \alpha<0\end{cases}$

2.1.2. Power integral of first species

The integral $\int_{a}^{+\infty} \frac{dx}{x^p}$ where p is a constant and a>0 is said a power integral. We can easily prove that $\int_{a}^{+\infty} \frac{dx}{x^p}=\begin{cases}\text{convergent} & \text{if} \quad p>1\\ \text{divergent} & \text{if} \quad p<1\end{cases}$

2.2. Convergence criteria for first species generalized integrals

The following criteria are given when one of the boundaries is +∞. Similar criteria are given when one of the boundaries is -∞. We shall suppose that f(x) is continuous ➔ integrable on any finite interval [a, b].

2.2.1. Comparison criteria

A) CONVERGENCE

Let's consider g(x), such that 0 ≤ f(x) ≤ g(x) for every x ≥ a.

So if $\int_{a}^{+\infty} g(x)dx$ is convergent, $\int_{a}^{+\infty} f(x)dx$ is also convergent.

B) DIVERGENCE

Let's consider g(x), such that $0 \leq g(x) \leq f(x)$ for every $x \geq a$.

So if $\int_a^{+\infty} g(x)dx$ is divergent, $\int_a^{+\infty} f(x)dx$ is also divergent.

2.2.2. Quotient criteria

If $f(x) \geq 0$ and $g(x) \geq 0$ for every $x \geq a$, and if $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L$

1. If $L \neq 0$ and $L \neq +\infty$ so $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$ are of the same nature.

2. If $L = 0$. So, if the integral $\int_a^{+\infty} g(x)dx$ is convergent, $\int_a^{+\infty} f(x)dx$ converges also.

3. If $L = +\infty$. So if the integral $\int_a^{+\infty} g(x)dx$ is divergent, $\int_a^{+\infty} f(x)dx$ diverges also

2.2.3. Theorem 1(convergence)

Let $\lim_{x\to+\infty} x^p f(x) = L$. So:

1. $\int_a^{+\infty} f(x)dx$ converges if $p > 1$ and if L is finite

2. $\int_a^{+\infty} f(x)dx$ diverges if $p \leq 1$ and if $L \neq 0$(L can be non defined).

REMARK

Similar criteria can be proposed if the comparison function is $g(x) = e^{-tx}$.

2.3. Absolutely convergent integrals

The integral $\int_a^{+\infty} f(x)dx$ is said absolutely convergent, if $\int_a^{+\infty} |f(x)|dx$ converges.

If $\int_a^{+\infty} f(x)dx$ converges but $\int_a^{+\infty} |f(x)|dx$ diverges, $\int_a^{+\infty} f(x)dx$ is then semi-convergent.

Theorem 2

Every integral that is absolutely convergent is convergent.

3. Generalized integrals of second species

Definition 1

Let f(x) a non bounded function only at point a of the finite interval [a, b]. So by definition:

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

If this limit exists the integral is said convergent, if it does not exist, it is said divergent (or senseless)

Definition 2

Similarly, by definition

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

The integral is said convergent or divergent, depending on the existence of the limit.

REMARK

If f(x) is non-bounded only at a point c of the interval [a, b], so by definition:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

We can extend this definition to cases where f(x) is non-bounded at two or multiple points of the interval [a, b].

3.1. Generalized integral of particular functions

1. The integral $\int_a^b \frac{dx}{(x-a)^p}$ where p is a constant is said power integral of second species. It converges if p < 1 and diverges if p ≥ 1
2. Similarly $\int_a^b \frac{dx}{(b-x)^p}$ converges if p < 1 and diverges if p ≥ 1

3.2. Convergence criteria

The following criteria are given whenever f(x) is non-bounded only at point a of the interval [a, b]. Similar criteria exist if f(x) is non-bounded only at point b of the interval [a, b].

3.2.1. Comparison criteria

A) CONVERGENCE

Let's consider a function g(x) such that 0 ≤ f(x) ≤ g(x) for every $x \in [a,b]$.

So if $\int_a^b g(x)dx$ is convergent, $\int_a^b f(x)dx$ is convergent also.

EXAMPLE $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ converges because $\int_1^5 \frac{dx}{\sqrt{x-1}}$ and $0 < \frac{1}{\sqrt{x^4-1}} < \frac{1}{\sqrt{x-1}}; \forall x > 1$

B) DIVERGENCE

Let's consider a function g(x) such that 0 ≤ g(x) ≤ f(x) for every $x \in [a,b]$.

So if $\int_a^b g(x)dx$ is divergent, $\int_a^b f(x)dx$ is divergent also.

EXAMPLE $\int_3^6 \frac{Logx}{(x-3)^4}dx$ diverges because $\int_3^6 \frac{dx}{(x-3)^4}$ diverges and

$\frac{Logx}{(x-3)^4} > \frac{1}{(x-3)^4}; \forall x > 0$

3.2.2. Quotient criteria

If f(x) ≥ 0 and g(x) ≥ 0 for every x ≥ a, and if $\lim_{x \to a} \frac{f(x)}{g(x)} = L$

1. If L ≠ 0 and L ≠ +∝ So $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are of the same nature.
2. If L = 0 So, if the integral $\int_a^b g(x)dx$ is convergent, $\int_a^b f(x)dx$ converges also.
3. If L = +∝ So if the integral $\int_a^b g(x)dx$ is divergent, $\int_a^b f(x)dx$ diverges also.

Theorem 3

Let $\lim_{x \to a^+} (x-a)^p f(x) = L$. So:

1. $\int_a^b f(x)dx$ converges if p < 1 and if L is finite
2. $\int_a^b f(x)dx$ diverges if p ≥ 1 and if L ≠ 0(L can be infinite).

EXAMPLE 1 $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ converges because $\lim_{x\to 1^+}(x-1)^{1/2} f(x) = \frac{1}{2}$

EXAMPLE 2 $\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$ diverges because $\lim_{x\to 3^-}(3-x) f(x) = \frac{1}{\sqrt{10}}$

3.3. Absolutely convergent integrals

The integral $\int_a^b f(x)dx$ is said absolutely convergent, if $\int_a^b |f(x)|dx$ converges.

If $\int_a^b f(x)dx$ converges but $\int_a^b |f(x)|dx$ diverges, $\int_a^b f(x)dx$ is then said semi-convergent.

Theorem 4

Every absolutely convergent integral is convergent.

4. Generalized integrals of third species

The improper integrals of third species can be written as functions of the first two species, so the correspondent criteria can be applied.
