Chapter 1

Vectors

Part1

1. Definition of points in space

1.1 Points and vectors

A number can be represented by a point on a line

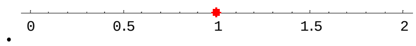


Figure 1

A pair of numbers (x, y) can be used to represent a point in a plane.

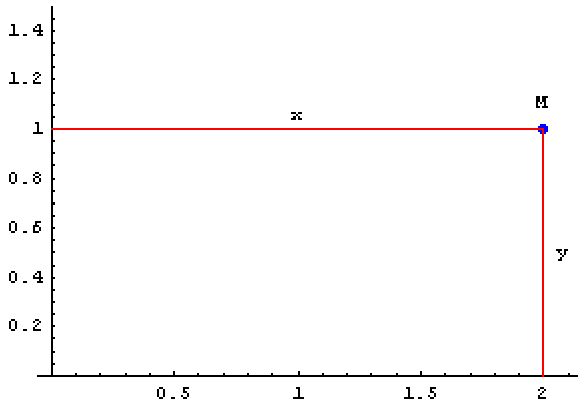


Figure 2

A triple of numbers (x, y, z) can be used to represent a point in space.

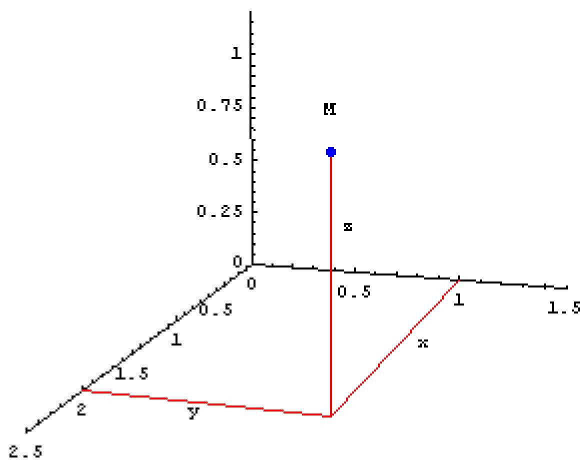


Figure 3

The point definition can be generalized into the following form $(x_1, x_2, ..., x_n) \in R^n$ even thought it can not be represented geometrically.

1.2 Addition in R^n

Let's consider $A = (a_1, a_2, ..., a_n)$ and $B = (b_1, b_2, ..., b_n)$ two points of R^n. We define the sum of these two points by the following: $A + B = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$

Let $0 = (0, 0, ..., 0)$, $-A = (-a_1, -a_2, ..., -a_n)$, prove the following:

$A + B = B + A$ $(A + B) + C = A + (B + C)$

$0 + A = A + 0$ $A + (-A) = 0$

1.3 Multiplication of a point by a number

Let $A = (a_1, a_2, ..., a_n) \in R^n$ and $\lambda \in R$. We define: $\lambda A = (\lambda a_1, \lambda a_2, ..., \lambda a_n)$

Prove the following properties

- $\lambda(A + B) = \lambda A + \lambda B$
- $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$
- $(-1)\times A = -A$

Example

- $A = (2, 9); B = (4, 5) \Rightarrow A+B = (6, 14)$
- $A = -(3,-4); B = (-3, 8) \Rightarrow A+B = (-6, 12)$

2. Located vectors

2.1 Located vectors

We define a located vector to be an ordered pair of points A and B of R^n which we write $\overrightarrow{AB}$. We visualize this as an arrow between A and B.

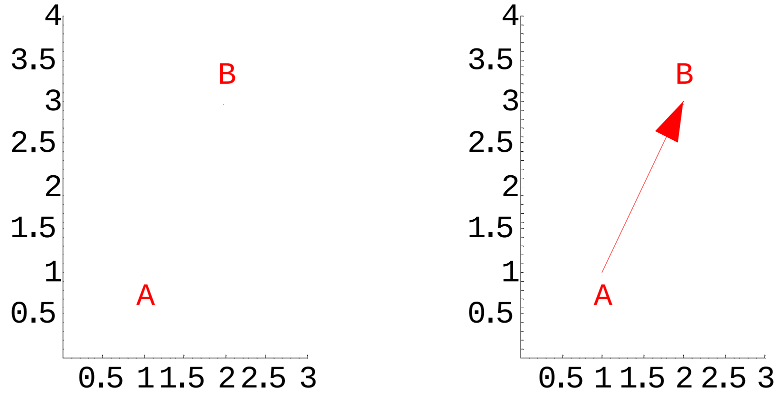


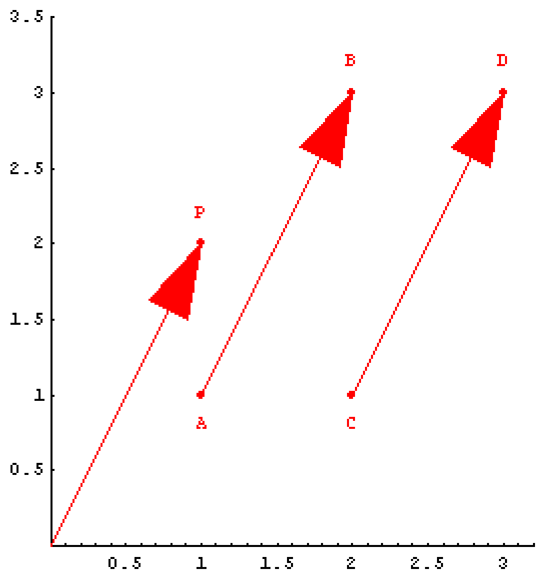
Figure 4

2.2 Equivalent vectors

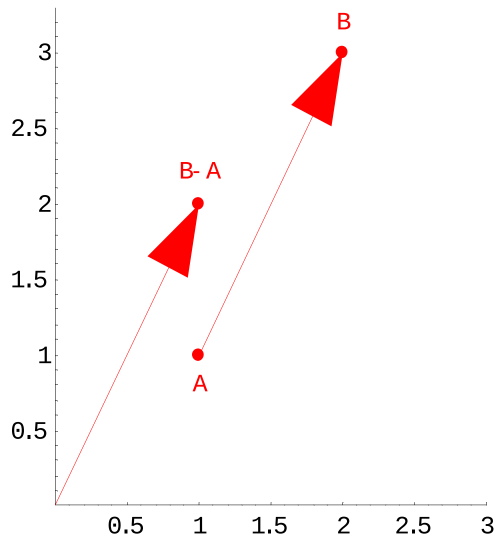
Two located vectors (A,B) and (C,D) are equivalent if :

B - A = D - C

A located vector (O,P) starting from the origin O is entirely determined by its end point P(a,b). A "bijection" is an application in R^n that associates for a giver vector (A,B) a vector located at the origin.



Remarque: The vector B-A is a vector having O as an origine and B-A as an extremity



2.3 Parallel vectors

Two located vectors (A,B) et (C,D) are said parallel if there is a number k≠0 such that:

B - A = k (D - C)

If k> 0 we say that they have the same direction

If k< 0 we say that they have opposite directions

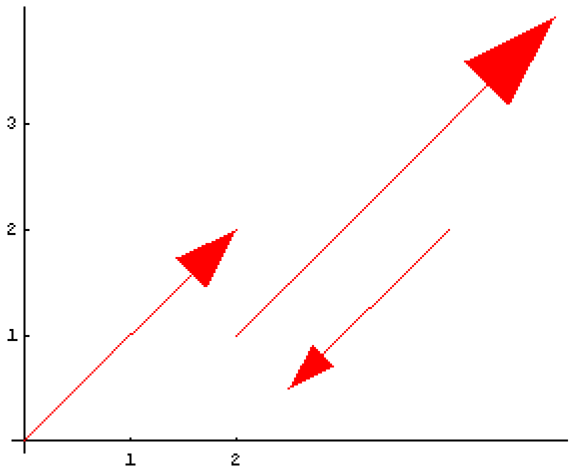


Figure 5

2.4 Scalar product

2.4.1 Scalar product

Let $A = (a_1, a_2, ..., a_n)$ and $B = (b_1, b_2, ..., b_n)$ two points of R^n. We define their scalar product to be:

$A \cdot B = a_1 b_1 + a_2 b_2 + ... + a_n b_n$

Provec:

- A . B = B . A
- A . (B + C) = A . B + A . C
- $(\lambda A).B = \lambda (A . B) = A.(\lambda B)$

2.4.2 Perpendicular vectors

Two vectors A and B are said perpendicular if and only if A . B = 0

2.5 Length of a vector

2.5.1 Length of a vector

The length of a vector A, $\|A\|$, is the real positive number defined by

$$\|A\| = \sqrt{A.A} = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$$

Prove

$$\|A\| = 0 \Rightarrow A = 0$$

$$\|\lambda A\| = |\lambda|\|A\|$$

$$\|A + B\| \leq \|A\| + \|B\|$$

2.5.2 Distance

The distance between two vectors A and B is defined by

$$\|A - B\| = \sqrt{(A - B)(A - B)}$$

2.5.3 Circle and disc

The equation of a circle, centered in P and having a radius a>0 is given by :

$$\|X - P\| = r$$

If $X = (x, y)$ and $P = (a, b) \Rightarrow$ The circle equation would be:

$$\|X - P\| = r \Rightarrow \sqrt{\|(x, y) - (a, b)\|} = \sqrt{\|(x - a, y - b)\|} = \sqrt{(x - a)^2 + (y - b)^2}$$

The equation of a disc, centered in P and having a radius a>0, is $\|X - P\| < a$

2.6 Orthogonal projection

Let A and B be two located vectors of R^n; $B \neq 0$. The orthogonal projection of A over B is vector P that verifies the following:

$$(A - P).B = 0$$

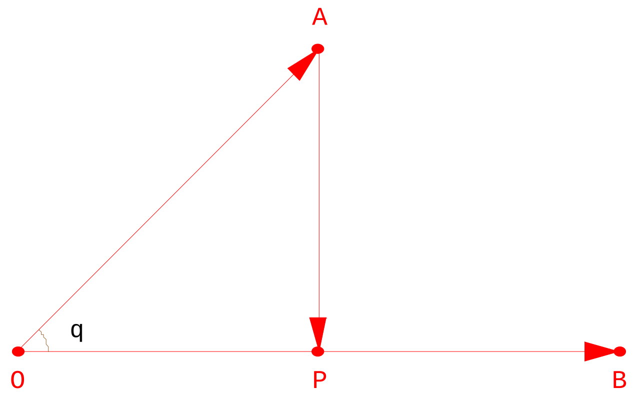


Figure 6

COMPUTATION OF K

$$(A-P).B = 0 \Rightarrow A.B = P.B = kB.B \Rightarrow k = \frac{A.B}{B.B}$$

Let $\theta = \widehat{AOB}$

$$\cos\theta = \frac{\|P\|}{\|A\|} = \frac{\sqrt{P.P}}{\sqrt{A.A}} = \frac{\sqrt{k^2\ B.B}}{\sqrt{A.A}} = k\times\frac{\sqrt{B.B}}{\sqrt{A.A}} = \frac{A.B}{B.B}\times\frac{\sqrt{B.B}}{\sqrt{A.A}} = \frac{A.B}{\|A\|\ \|B\|}$$

$$\Rightarrow A.B = \|A\|\,\|B\|\cos\theta$$

Conclusion: $k = \frac{A.B}{B.B}$ et $A.B = \|A\|\,\|B\|\cos\theta$

2.7 Parametric lines

We define the parametric equation (or representation) of a straight line passing through a point P in the direction of a vector A≠0 to be:

M = P + t A

Where t runs through all numbers

A line passing through two points A and B has the following equation:

M = A + t (B - A) or M = A + t (A - B)

The direction of this line is vector A - B or vector B – A

Whenever $0 \le t \le 1$, M = A + t (B - A) is the equation of segment [A,B]

2.8 Planes

We can describe planes in space by an equation analogous to the equation of the line. Let A be a point in space and N a located vector. We define the plane passing through P and perpendicular to N by the following equation:

N. (X - A) = 0 or N.X = N.A

Part 2

3. Differentiation of vectors

3.1 Parametric curve

Let I = [a, b] of R. The function that associates for every point t of I a located vector X(t) of R^n forms a parametric curve.

$$I \longrightarrow R^n;$$

$$t \longrightarrow X(t)$$

3.2 Speed and velocity vector

If X(t) is a parametric curve, vector X'(t) is said velocity vector at time t. It's a vector located at the origin, but when we translate it to the point X(t), we visualize it as a tangent to the curve.

The quantity $v(t) = \|X'(t)\|$ is said velocity at time t.

3.3 Acceleration vector

If X(t) is a parametric curve, vector X''(t) is said acceleration vector at time t. The quantity $a(t) = \|X''(t)\|$ is said acceleration at time t.

3.4 Derivation rules

$$(X(t)+Y(t))' = X'(t)+Y'(t) \qquad (\lambda X(t))' = \lambda X'(t)$$

$$(X(t).Y(t))' = X'(t).Y(t)+X(t).Y'(t) \qquad (f(t)Y(t))' = f'(t)Y(t)+f(t)Y'(t)$$

4. Length of curves

By definition the length L of a curve X(t) is:

$$L = \int_a^b ds = \int_a^b v(t)dt = \int_a^b \|X'(t)\| dt$$

We can note $ds = \|X'(t)\| dt$

4.1 Curve given by Cartesian coordinates

$$L = \int_a^b ds = \int_a^b \|X'(t)\| dt = \int_a^b \sqrt{x_1'^2 + x_2'^2 + \ldots + x_n'^2}\, dt$$

Example 1

Length of a semi-circle $y=\sqrt{1-x^2}$

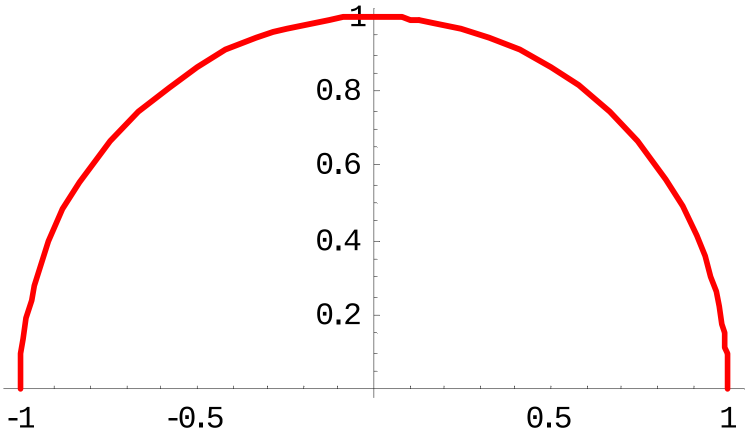


Figure 7

$$L=\int_{-1}^{+1}\sqrt{1+y'^2(x)}dx=\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}=2\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}=2\left(Arc\sin 1-Arc\sin 0\right)=2\frac{\pi}{2}=\pi$$

4.2 Curve given by Polar coordinates

DEFINITION

$x=r\cos\theta$

$y=r\sin\theta$

$0\leq\theta\leq 2\pi$

REPRESENTATION

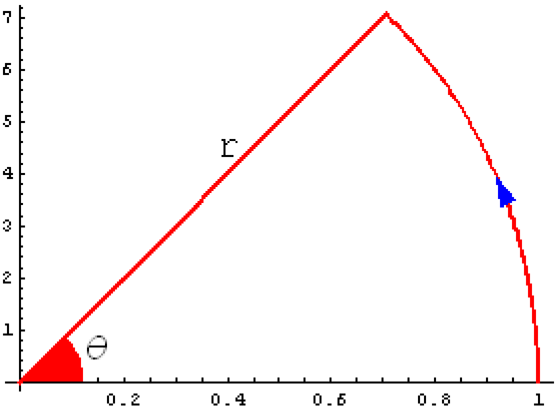


Figure 8

$$L=\int_a^b ds=\int_a^b\left\|X'(t)\right\|dt=\int_a^b\sqrt{r'^2(t)+r^2(t)\theta'^2(t)}dt$$

4.3 Curve given by Cylindrical coordinates

DEFINITION

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$0 \le \theta \le 2\pi$$

REPRESENTATION

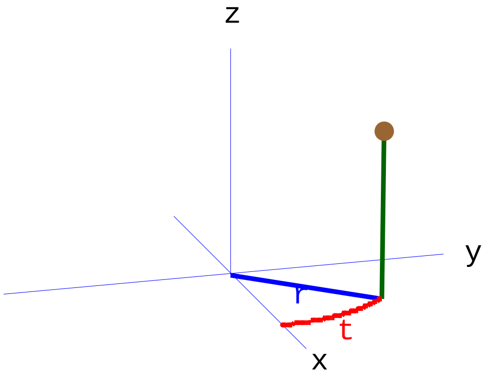


Figure 9

$$L = \int_a^b \|X'(t)\| dt = \int_a^b \sqrt{r'^2(t) + r^2(t)\theta'^2(t) + z'^2(t)}\,dt$$

4.4 Curve give by spherical coordinates

DEFINITION

$$x = \rho \sin\varphi \cos\theta;$$
$$y = \rho \sin\varphi \sin\theta;$$
$$z = \rho \cos\varphi$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$
$$0 \le \rho$$

REPRESENTATION

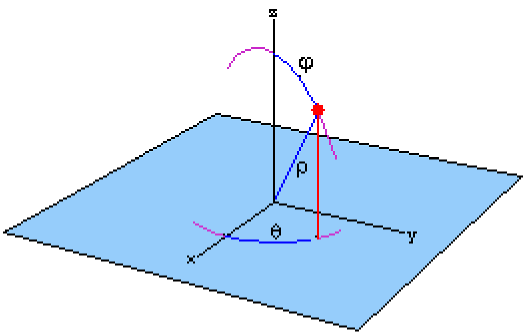


Figure 10

$$L = \int_a^b \|X'(t)\| dt = \int_a^b \sqrt{\rho'^2(t) + \rho^2(t)\varphi'^2(t) + \rho^2(t)\theta'^2(t)\sin^2\varphi(t)}\,dt$$