Chapter II

Functions of several variables

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Part 1

1. Definitions

1.1 Norms in R^n

We know that we can use in R^n one of the following three equivalent norms:

$$N_1(X) = \sum_{i=1}^{n} |x_i| \quad ;$$

$$N_2(X) = \sqrt{x_1^{\,2} + x_2^{\,2} + \ldots x_n^{\,2}} \quad ;$$

$$N_3(X) = \sup_{1 \le i \le n} |x_i|$$

We will consider R^n fitted with the norm N_2, said Euclidian norm, which we will note:

$$\forall X \in R^n \qquad \|X\| = \sqrt{x_1^{\,2} + x_2^{\,2} + \ldots x_n^{\,2}}$$

1.2 Numerical functions

Let A be a set of points in the R^n space. A numerical function is an association which to each of element of S associates a number.

$$\begin{array}{cccc} f: & A & \rightarrow & R \\ (x_1, x_2, \ldots, x_n) & & \rightarrow & f(x_1, x_2, \ldots, x_n) \end{array}$$

EXAMPLE

$$f(x,y) = x^2 + y^2 \quad f: R^2 \rightarrow R$$

$$f(x,y,z) = x^y \quad f: R^{*+} \times R^2 \rightarrow R$$

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad f: R^2 - \{(0,0)\} \rightarrow R$$

1.3 Graph of a function

Let f be an n variables numerical function defined on A $\subset R^n$. The set of points in (n+1)-space of the form:

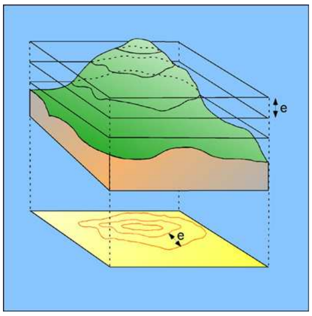
$$(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$$

Is called graph of f, the $(x_1, x_2, \ldots, x_n)$ being in the domain of definition of f.

1.4 Surface and level curve

> Surfaces and level curves are described by equations
> $f(x_1, x_2, \ldots, x_n) = cte$

If f is function of 2 variables, a level curve represents the intersection of the surface with the plan z = constant.



EXAMPLE 1

$$z = f(x, y) = \sqrt{4 - x^2 - y^2} = 1 \Rightarrow x^2 + y^2 = 3$$

Is the equation of the circle formed by the intersection of the **sphere** with the plan z=1.

This equation represents also the projection of this intersection on the plan xOy.

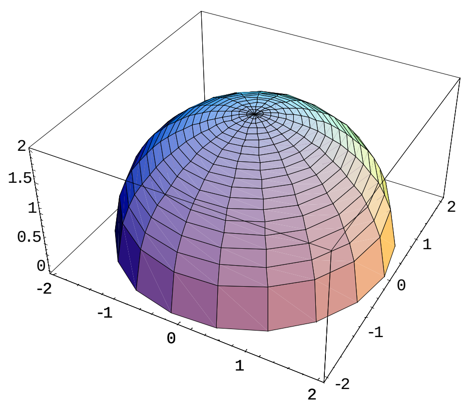


Figure 1

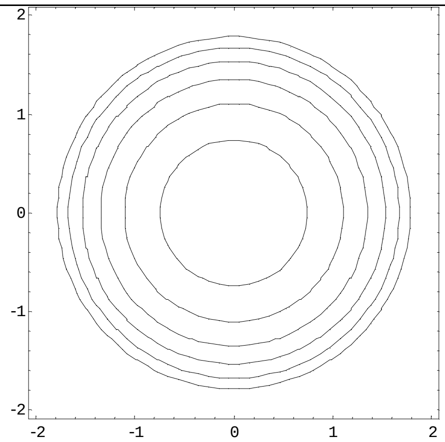


Figure 2

EXAMPLE 2

$$f(x,y)=e^{-\frac{x^2+y^2}{8}}\left(\cos x^2+\sin y^2\right)$$

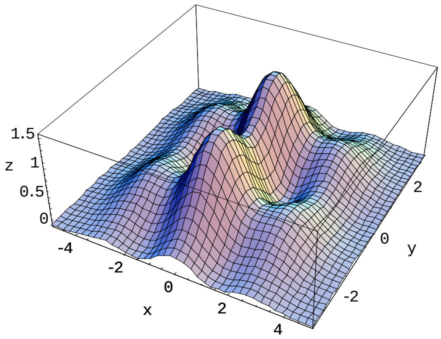


Figure 3

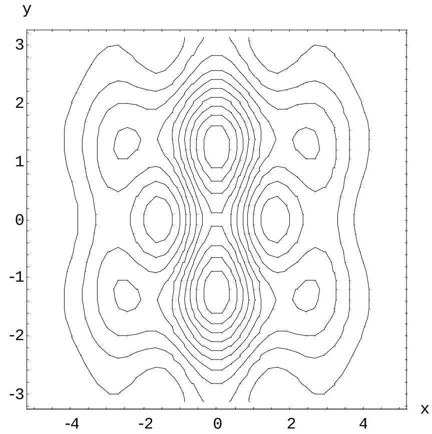


Figure 4

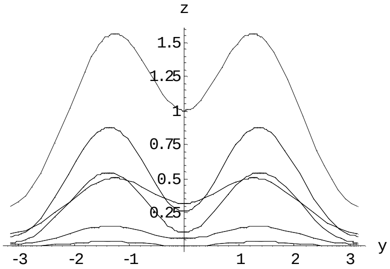


Figure 5

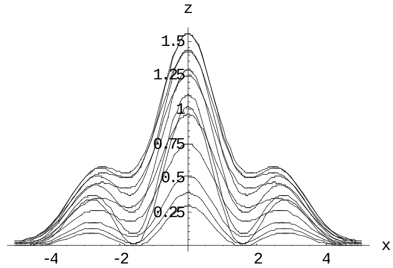


Figure 6

2. Limit and continuity

2.1 Limit of a function at a selected point.

Let consider f an n variables numerical function defined on a part A of R^n.

Let $X=(x_1,x_2,...,x_n)$

It's said that f(X) strives towards a limit l when $X=(x_1,x_2,...,x_n)$ strives toward $a=(a_1,a_2,...,a_n)$ if and only if :

$\forall\varepsilon>0\ \ \exists\alpha>0$ such that $\|X-a\|<\alpha\Rightarrow|f(X)-l|<\varepsilon$

In other words

$$\forall\varepsilon>0\ \ \exists\alpha>0 \text{ such that } \sqrt{(x_1-a_1)^2+(x_2-a_2)^2+...+(x_n-a_n)^2}<\alpha\Rightarrow|f(X)-l|<\varepsilon$$

We shall write: $\lim_{X\to a} f(X)=l$

2.2 Continuity

Let consider f an n variables numerical function defined on a part A of R^n.

It's said that f is continue at point $a=(a_1,a_2,...,a_n)\in R^n$ if and only if

$$\lim_{X \to a} f(X) = f(a)$$

2.3 Examples

Do the following functions have a limit when (x, y) tends towards (0,0)?

$$f(x,y) = \frac{1}{x^2+y^2}; \quad f(x,y) = \frac{x^2-y^2}{x^2+y^2}; \quad f(x,y) = \frac{x^3y^3}{x^2+y^2}$$

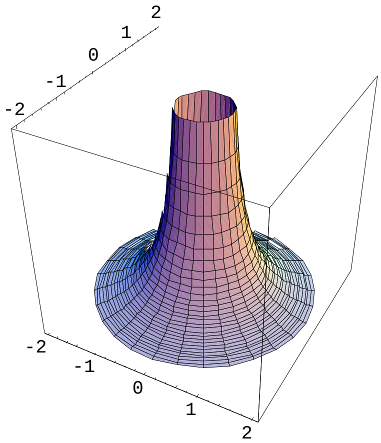


Figure 7

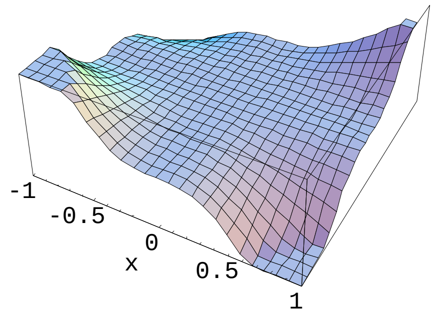


Figure 8

2.4 Cauchy criteria

Let A $\subseteq R^n$ and f: A $\rightarrow$ R. Let a, a point belonging to A. The following conditions are equivalent:

(a) $\lim_{X \to a} f(X) = l$

(b) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall (X,Y) \in A \times A$ et $\|X - a\| < \delta, \|Y - a\| < \delta \Rightarrow |f(X) - f(Y)| < \varepsilon$

In fact:

Knowing that: $X = (x_1, x_2, ..., x_n)$; $Y = (y_1, y_2, ..., y_n)$, and $a = (a_1, a_2, ..., a_n)$

Let's demonstrate that (a) $\Rightarrow$ (b)

$$\lim_{X \to a} f(X) = l \Leftrightarrow \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that } \|X - a\| < \delta \Rightarrow |f(X) - l| < \varepsilon$$

In the same manner

$$\lim_{X \to a} f(X) = l \Leftrightarrow \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that } \|Y - a\| < \delta \Rightarrow |f(Y) - l| < \varepsilon$$

So

$$\lim_{X \to a} f(X) = l \Leftrightarrow \forall 2\varepsilon > 0 \;\; \exists \delta > 0 \text{ such that } \|X - a\| < \delta \quad \text{and } \|Y - a\| < \delta$$

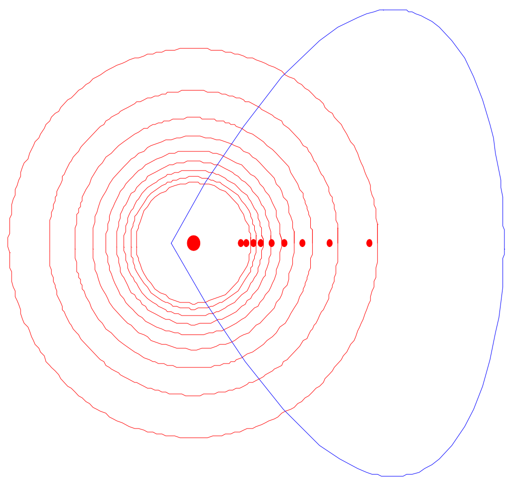
$$\Rightarrow |\text{f(X)-f(Y)}| < |f(X) - l| + |f(Y) - l| < 2\varepsilon$$

(what needed to be demonstrated)

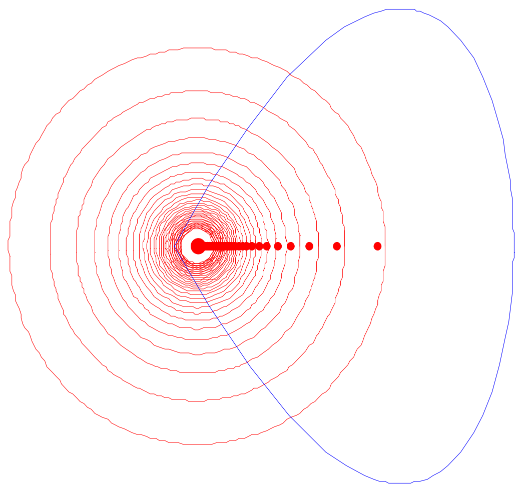
Let's demonstrate that (b) $\Rightarrow$ (a)

Let's consider that (b) is verified, a being a point belonging to A

$$\forall B(a, \frac{1}{n}) \;\Rightarrow\; B(a, \frac{1}{n}) \cap A \neq \phi \Rightarrow \exists\, X_n \in A \text{ such that } \|X_n - a\| < \frac{1}{n}.$$



This series tends towards a. Let's choose N in a manner that $\frac{1}{N} < \frac{\delta}{2}$. So we have:



$$\forall m, n > N \;\; \|X_n - X_m\| \leq \|X_n - a\| + \|X_m - a\| < \frac{\delta}{2} + \frac{\delta}{2} < \delta$$

$$\Rightarrow |f(X_n) - f(X_m)| < \varepsilon$$

So the f(X_n) series is a convergent Cauchy series in R. Let l be its limit. Let's show that l is also the limit of f(X) when X $\rightarrow$ a. We shall demonstrate that:

$$\forall \varepsilon > 0 \quad \exists \delta_1 > 0 \text{ such that } \|X-a\| < \delta_1 \quad \Rightarrow |f(X) - l| < \varepsilon$$

Since $\lim\limits_{n\to+\infty} X_n = a$ we have:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that } \|X_n-a\| < \delta \quad \Rightarrow |f(X_n) - l| < \varepsilon$$

Let's consider $\delta_1 = \dfrac{\delta}{2}$

$$\|X - X_n\| \leq \|X - a\| + \|X_n - a\| < \delta \overset{from(b)}{\Rightarrow} |f(X) - f(X_n)| < \varepsilon$$

So

$|f(X) - l| < |f(X) - f(X_n)| + |f(X_n) - l| < \varepsilon + \varepsilon < 2\varepsilon$ **(what needed to be demonstrated)**

EXAMPLES

- $\begin{cases} f(x,y) = \dfrac{x^3 + y^3}{x^2 + y^2} \\ f(0,0) = 0 \end{cases}$

  Continuous at the origin

- $\begin{cases} g(x,y) = \dfrac{xy}{x^2 + y^2} \\ g(0,0) = 0 \end{cases}$

  Is not continuous at the origin.

2.5 Lemme

> If an n variables function is continuous in a point $a = (a_1, a_2, ..., a_n)$, so it's continuous for every one of its variables x_i in this point.

So $\lim\limits_{x_i \to a_i} f(X) = f(a)$

REMARQUE 7

The reciprocal use of this lemme is an error.

EXAMPLE

$$\begin{cases} f(x,y) = \dfrac{xy^3 + x^3 y}{x^4 + y^4} \\ f(0,0) = 0 \end{cases}$$

This function is continuous for every one of its variables at (0,0) because f(x,0) and f(0,y) are 2 constant functions ➔ continuous functions. However f is not continuous at (0,0).

$$f(my,y)=\frac{my^4+m^3y^4}{m^4y^4+y^4}=\frac{m+m^3}{m^4+1}\Rightarrow \lim_{y\to 0} f(my,y)\neq 0=f(0,0)$$

Part 2

3. Partial derivatives, gradient

3.1 Partial derivative

Let $f(x_1,x_2,\dots,x_n)$ an n variables function defined on a part A of R^n. We define the partial derivative of f according to the variable xi by the following expression

$$f'_{x_i}(X)=D_i f(X)$$
$$=\frac{\partial\ f(X)}{\partial\ x_i}$$
$$=\lim_{h\to 0}\frac{f(x_1,x_2,\dots,x_i+h,\dots,x_n)-f(x_1,x_2,\dots,x_i,\dots,x_n)}{h}$$
(0.1)

EXAMPLE 1

$f(x,y)=x^2y^3$; $f(x,y)=x\cos(xy)$;

$f(x,y)=e^{xy}$; $f(x,y,z)=x^2y\cos(yz)$

EXAMPLE 2

$$f(x,y)=\frac{x^3+y^3}{x^2+y^2}\ pour\ (x,y)\neq(0,0)\ et\ f(0,0)=0.$$

Compute $f'_x(0,0)$.

IN FACT

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

3.2 Gradient

Let f an n variables function defined on a part A of R^n . The gradient of f is the vector defined by:

$$gradf(X)=(f'_{x_1}(X),f'_{x_2}(X),\dots,f'_{x_n}(X))$$

Demonstrate:

- grad (f+g) = grad f + grad g

- grad (λ f) = λ grad f

EXAMPLES

$$e^{xyz} \ ; \ Ln(z+\sin(y^2-x)) \ ; \ e^{-2z} \ Cos(yz)$$

4. Differentiability and gradient

4.1 Introduction

Let f an n variables function (n ≥ 1), defined on an open set U of R^n. For all vectors H such that $\|H\|$ is small (and $H \neq 0$), the point X+H also lies in the open set U. However we can not form a quotient

$$\frac{f(X+H)-f(X)}{H}$$

because it's meaningless to divide a vector by another vector. In order to define what we mean for a function f to be differentiable, we must therefore find a way which does not involve dividing by H.

We reconsider the case of one variable. Let us fix a number x. We had defined the derivative to be:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

We can write

$$\Phi(h) = \frac{f(x+h)-f(x)}{h} - f'(x)$$

$$\Rightarrow \lim_{h \to 0} \Phi(h) = 0$$

$$\Rightarrow f(x+h)-f(x) = f'(x)\cdot h + h\Phi(h) \quad \text{for } h \neq 0$$

Let $\Phi(0)=0$, We will therefore get the following relation:

$$f(x+h)-f(x) = f'(x)\cdot h + |h|\Phi(h) \quad \forall \text{ h} \quad \text{with} \quad \lim_{h \to 0} \Phi(h) = 0 \qquad (\alpha)$$

Conversely, suppose that there exists a number a and a function g(h) such that:

$$f(x+h)-f(x) = a\cdot h + |h| g(h) \quad \forall \text{ h} \quad \text{with} \quad \lim_{h \to 0} g(h) = 0$$

$$\Rightarrow \qquad \frac{f(x+h)-f(x)}{h} = a + \frac{|h|}{h} g(h) \quad \text{with } h \neq 0$$

Taking the llimit as h approaches 0 $\lim_{h\to 0}\frac{|h|}{h}g(h)=0$, we observe that $a=f'(x)$

CONCLUSION

The existence of a number a and a function g satisfying the above could have been used as the definition of differentiability in the case of functions of one variable. The great advantage is that no h appears in the denominator. This relation will suggest to us how to define differentiability for functions of several variables and how to prove the chain rule for them.

4.2 Definition (Differentiability)

We say that f is differentiable at point X if its partial derivatives exists and if there exists a function g(H) such that $\lim_{H\to O} g(H)=0$ and

$$f(X+H)-f(X)=gradf(X).H+\|H\|g(H)$$

For example, if f is function of 2 variables (x,y), and the vector H (h, k). The differentiability condition becomes:

$$f(x+h,y+k)-f(x,y)=f_x^{'}(x,y)\cdot h+f_y^{'}(x,y)\cdot k+\sqrt{h^2+k^2}\,g(h,k)$$

with $\lim_{H\to O} g(h,k)=0$

EXAMPLES

Study the differentiability of f at point (0, 0)

$$f(x,y)=(x^2+y^2)\sin\frac{1}{x^2+y^2}\quad \text{if}\quad (x,y)\neq(0,0)\quad et\quad f(0,0)=0$$

4.2.1 Theorem 1

Every function differentiable at a point X is continuous at this point.

In fact:

When H $\to$ 0 grad f(X).H $\to$0. $\Rightarrow$ f(X+H) $\to$ f(X) $\Rightarrow$ continuity.

4.2.2 Theorem 2

Let f an n variables numeric function defined on a part A of R^n. If the partial derivatives of f exist and are continuous in a point, then f is differentiable.

Let's demonstrate this theorem for a 2 variables function.

Let A=(a, b) and H=(h, k). We have

$$\begin{aligned} f(A+H)-f(A) &= f(a+h,b+k)-f(a,b)\\ &= f(a+h,b+k)+f(a,b+k)-f(a,b+k)-f(a,b)\end{aligned}$$

The function $f\left(a,y\right)$ is a single variable "y" differentiable function. By applying the finite increase theorem we get:

$$\exists\beta\in\left]0,1\right[\ /\ f(a,b+k)-f(a,b)=k\ f_y^{'}(a,b+\beta k)$$

By doing the same to the first variable we get:

$$\exists\alpha\in\left]0,1\right[\ /\ f\left(a+h,b+k\right)-f\left(a,b+k\right)=hf\,'_x\left(a+\alpha h,b+k\right)$$

So

$$f(A+H)-f(A)=hf_x^{'}(a+\alpha h,b+k)+kf_y^{'}(a,b+\beta k)$$

Finally, using:

$$\Gamma_1\left(H\right)=f\,'_x\left(a+\alpha h,b+k\right)-f\,'_x\left(a,b\right)$$
$$\Gamma_2\left(H\right)=f\,'_y\left(a,b+\beta k\right)-f\,'_y\left(a,b\right)$$

We get:

$$h\,\Gamma_1\left(H\right)=h\,f\,'_x\left(a+\alpha h,b+k\right)-h\,f\,'_x\left(a,b\right)$$
$$k\,\Gamma_2\left(H\right)=k\,f\,'_y\left(a,b+\beta k\right)-k\,f\,'_y\,a,b$$

$$h\Gamma_1(H)+k\Gamma_2(H)=hf\,'_x\left(a+\alpha h,b+k\right)+kf\,'_y\left(a,b+\beta k\right)-\left(hf\,'_x\left(a,b\right)+kf\,'_y\left(a,b\right)\right)$$
$$h\Gamma_1(H)+k\Gamma_2(H)=f(A+H)-f(A)-\left(hf\,'_x\left(a,b\right)+kf\,'_y\left(a,b\right)\right)$$

$$f(A+H)-f(A)=hf_x^{'}(a,b)+kf_y^{'}(a,b)+h\Gamma_1(H)+k\Gamma_2(H)$$

Since the partial derivatives are continuous, we deduce:

$$\lim_{H\to 0}\Gamma_1(H)=0 \quad \text{et} \quad \lim_{H\to 0}\Gamma_2(H)=0$$

Let:

$$h\Gamma_1(H)+k\Gamma_2(H)=\left\|H\right\|\Gamma(H)$$

$$\lim_{H\to 0}\Gamma(H)=\lim_{H\to 0}\frac{h\Gamma_1(H)+k\Gamma_2(H)}{\sqrt{h^2+k^2}}$$

$\Rightarrow$

$$\lim_{r\to 0}\Gamma^*(r,\theta)=\lim_{r\to 0}\frac{r\cos\theta\sum_1^*(r,\theta)+r\sin\theta\Gamma_2^*(r,\theta)}{\sqrt{r^2}}$$
$$=\lim_{r\to 0}[\cos\theta\Gamma_1^*(r,\theta)+\sin\theta\Gamma_2^*(r,\theta)]$$

But $\lim_{r\to 0}\Gamma_1^*(r,\theta)=0$ and $\lim_{r\to 0}\Gamma_2^*(r,\theta)=0$

$$\Rightarrow \quad \lim_{r\to 0}\Gamma^*(r,\theta)=0 \quad\Rightarrow\quad \lim_{H\to 0}\Gamma(H)=0$$

So, we have found a function $\Gamma(H)$ that satisfies the differentiability conditions:

$$f(A+H)-f(A)=hf_x^{'}(a,b)+kf_y^{'}(a,b)+\|H\|\Gamma(H)$$ **(what needed to be demonstrated)**

4.3 Definition (Differential)

if $f:R^2\to R$ is differentiable in a certain point (x_0,y_0). The differential of f at point (x_0,y_0) is defined by:

$$df(x_0,y_0)=f_x^{'}(x_0,y_0)dx+f_y^{'}(x_0,y_0)dy$$

For a 3 variables function:

$$df(x_0,y_0,z_0)=f_x^{'}(x_0,y_0,z_0)dx+f_y^{'}(x_0,y_0,z_0)dy+f_z^{'}(x_0,y_0,z_0)dz$$

This can be generalized for an n variables functions.

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5. Partial derivatives of superior order

5.1 Definition

Let f a 2 variables (x,y) function. The partial derivatives, when they exist, are also functions of (x,y). We can also compute their respective partial derivatives. So we get

$$D_1D_2f,\ D_2D_1f,\ D_1D_1f,\ D_2D_2f$$

that we shall write

$$D_1D_2f(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}(x,y)$$

and

$$D_1^2f(x,y)=D_1D_1f(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}(x,y)$$

5.1.1 Schwartz theorem

Let f be a function of 2 variables, defined on an open set U of 2-space. Assume that all the partial derivatives exists and are continuous. Then f''_{xy} and f''_{yx} are equal.

$$D_1D_2 f(x,y) = D_2D_1 f(x,y) \text{ or } \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

IN FACT

Let ω be a surrounding of point (0,0) such that for every (h,k) of ω, we have *(a+ h, b+ k)* in U. We define on ω the following function F:

$$F(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)$$

Let's take k as a constant and consider the following function:

$$\varphi(x) = f(x,b+k) - f(x,b)$$

We have :

$$\varphi(x+h) - \varphi(x) = f(x+h,b+k) - f(x+h,b) - f(x,b+k) + f(x,b) \Rightarrow$$

$$F(h,k) = \varphi(a+h) - \varphi(a)$$

This is a continuous and derivable function.

$$\varphi'(x) = f'_x(x,b+k) - f'_x(x,b)$$

From the finite increase theorem we can assume that there is an α between 0 and 1 such tha:

$$\varphi(x+h) - \varphi(x) = h\varphi'(x+\alpha h)$$

$$\Rightarrow \quad F(h,k) = h\left[f'_x(a+\alpha h,b+k) - f'_x(a+\alpha h,b)\right]$$

But the function $y \mapsto f'_x(a+\alpha h, y)$ is derivable. A new application of the finite increase theorem shows that there is a β between 0 and 1 such that:

$$F(h,k) = h\,k f''_{xy}(a+\alpha h,b+\beta k)$$

Since f''_{xy} is continuous at point (a,b) we have:

$$\lim_{(h,k)\to(0,0)} \frac{F(h,k)}{hk} = \lim_{(h,k)\to(0,0)} f''_{xy}(a+\alpha h,b+\beta k) = f''_{xy}(a,b)$$

Let Ψ be a single variable function:

$$\psi(y) = f(a+h, y) - f(a, y)$$

By thinking as previously, we find that:

$$\lim_{(h,k)\to(0,0)} \frac{F(h,k)}{hk} = \lim_{(h,k)\to(0,0)} f''_{yx}(a+\alpha h, b+\beta k) = f''_{yx}(a,b)$$

Which proves that

$$\lim_{(h,k)\to(0,0)} \frac{F(h,k)}{hk} = f''_{yx}(a,b) = f''_{xy}(a,b)$$ **(what needed to be demonstrated)**