Chapter III

Derivation of composite functions

Part 1

1. The chain rule

1.1. Definition of composite function

Let f be a function defined on some open set U. Let C(t) be a curve such that the values C(t) are contained in U. Then we can form the composite foC, which is a function of t, given by:

$$(f \circ C)(t) = f(C(t))$$

Reminder

when y = f(x) is a single variable function, and when x=x(t), we know that:
$f_t^{'}(x(t)) = f_x^{'}(x(t)) \cdot x_t^{'}(t)$. So we would like to find a common rule for multi-variable functions.

1.2. Chain rule (derivation of composite functions)

Let f be a multi-variable differentiable function. Let's consider X as a function of t:

$$X(t) = (x_1(t), x_2(t), ..., x_n(t))$$

So:

$$f_t^{'}(X(t)) = f_{x_1}^{'}(X(t)) \cdot x_1^{'}(t) + f_{x_2}^{'}(X(t)) \cdot x_2^{'}(t) + ... + f_{x_n}^{'}(X(t)) \cdot x_n^{'}(t)$$

That we can simply write:

$$f_t^{'} = f_{x_1}^{'} \cdot x_1^{'} + f_{x_2}^{'} \cdot x_2^{'} + ... + f_{x_n}^{'} \cdot x_n^{'}$$

In other terms:

$$\boxed{f_t^{'} = (f_{x_1}^{'}, f_{x_2}^{'}, ..., f_{x_n}^{'}).(x_1^{'}, x_2^{'}, ..., x_n^{'}) \Rightarrow f_t^{'} = grad\ f(X(t)).X_t^{'}(t)}$$

EXAMPLE:

$$f(x, y) = \sin xy$$

Let:

$$x = r\cos\theta,\ y = r\sin\theta$$

$$f_r^{'}(r\cos\theta, r\sin\theta) = gradf(x, y) \cdot (x_\theta^{'}, y_\theta^{'})$$
$$= f_x^{'}(r\cos\theta, r\sin\theta) \times x_\theta^{'} + f_y^{'}(r\cos\theta, r\sin\theta) \times y_\theta^{'}$$

1.3. Derivation of implicit functions

DEFINITION

> If several variables, 3 for example, are related as follow
>
> F(x, y, z)=0,
>
> One of the variables, z for example, is an implicit function of the other 2.

EXAMPLE

$(x^2+1)z - xy^2 = 0 \Rightarrow z = \varphi(x, y) = \dfrac{xy^2}{x^2+1}$ on □.

If F have partial derivatives we can demonstrate that, except for certain exceptional points, φ(x, y) has also partial derivatives.

Let's consider these 2 cases:

Case 1 : z=φ(x) defined by F(x, z)=0.

$$dF = 0 = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial z}dz$$

which leads to

$$\frac{dz}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \quad si \quad \frac{\partial F}{\partial z} \neq 0. \Rightarrow \frac{dz}{dx} = -\frac{F_x^{'}}{F_z^{'}}, \quad si \quad F_z^{'} \neq 0.$$

Case 2 : z=φ(x, y) defined by F(x, y, z)=0

$$dF = 0 = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

$$\Rightarrow \; dz = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}dx - \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}dy, \quad si \quad \frac{\partial F}{\partial z} \neq 0$$

$$\Rightarrow dz = -\frac{F_x^{'}}{F_z^{'}}dx - \frac{F_y^{'}}{F_z^{'}}dy, \quad si \quad F_z^{'} \neq 0$$

$$\Rightarrow \text{Since } dz = z_x^{'}dx + z_y^{'}dy \Rightarrow \text{if} \quad F_z^{'} \neq 0 \Rightarrow \begin{cases} z_x^{'} = -\dfrac{F_x^{'}}{F_z^{'}} \\ z_y^{'} = -\dfrac{F_y^{'}}{F_z^{'}} \end{cases}$$

2. Extremum of 2 variables numerical functions

2.1. Local extremum

Let f be a function defined in an open set U of R^2. $f : U \to R^2$

1. We say that f have a local maximum at P if and only if :

   $$\exists r > 0, \;\; \forall X \in U \cap B(P, r) \Rightarrow f(X) \leq f(P)$$

   In other terms, there exists a surrounding for P, where $f(P)$ is the maximum.

2. We say that f have a local minimum at P if and only if :

$$\exists r > 0,\ \ \forall X \in U \cap B(P,r) \Rightarrow f(P) \le f(X)$$

In other terms, there exists a surrounding for P, where $f(P)$ is the minimum.

3. We say that f have a local extremum at P, if and only if f have a local minimum or maximum at this point.

Example:

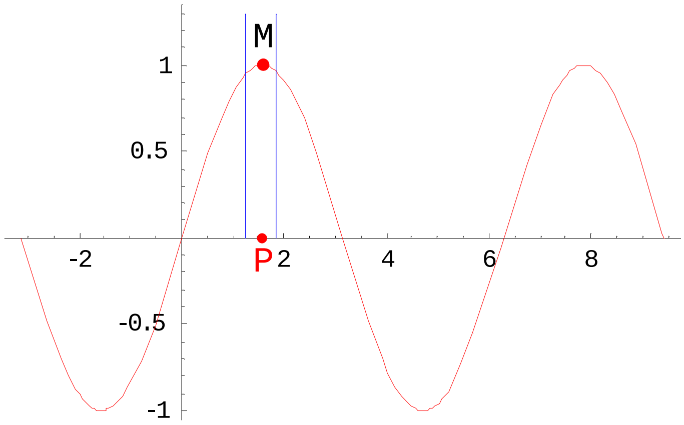

Figure 1

2.2. Critical points

If f is a differentiable function defined in an open set U. Let P be a point of U. If all the partial derivatives of f are null at P, we can say that is a critical point of f.

For a 2 variables function, point $P(x_0, y_o)$ is a critical point of f if and only if :

$$f_x^{'}(x_0, y_o) = 0 \text{ et } f_y^{'}(x_0, y_o) = 0$$

Theorem

Let f be a differentiable function defined in an open set U. Let P be a point where f have a local extremum. Subsequently, P is a critical point of f.

IN FACT

The demonstration is identical to the one used for a single variable function. Let's consider a non null vector H and a value of t, chosen in such a way that P+tH is always within U. Since $f(P)$ is maximum we have:

$$f(P + tH) \le f(P)$$

The single variable function $g(t) = f(P + tH)$ have a local maximum at point t=0, which means that $g'(0) = 0$

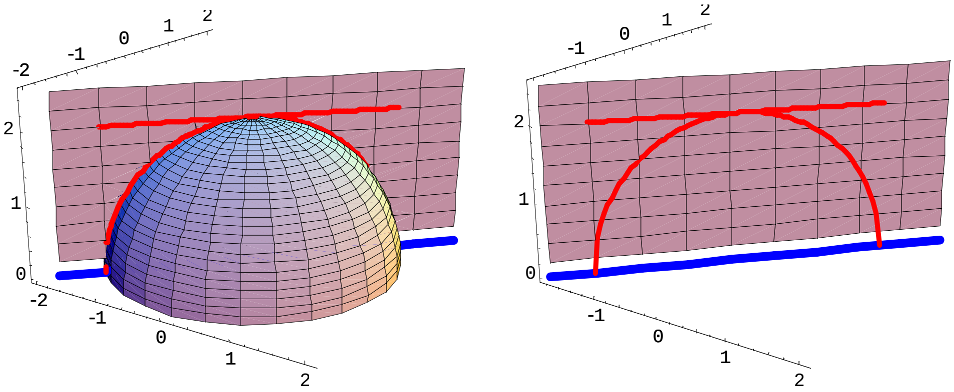

Figure 2

Or:

$$g'(t) = f_t^{'}(P+tH) = gradf\,(P+tH)\cdot(P+tH)' = gradf\,(P+tH)\cdot H$$

For t =0 : $g'(0) = gradf\,(P).H = 0$

This equality is true for every H so:

$$\Rightarrow gradf\,(P) = 0$$

$$\Rightarrow \left(f_x^{'}(P), f_y^{'}(P)\right) = (0,0)$$

$\Rightarrow$ P is a critical point of function *f*.

Here is a schematization of the tree cases of critical points: maximum, minimum and inflexion point

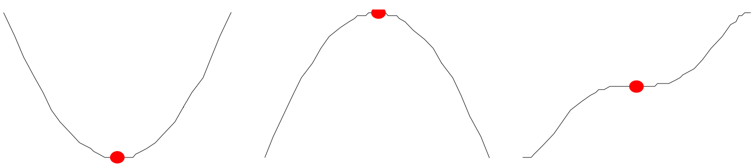

Figure 3

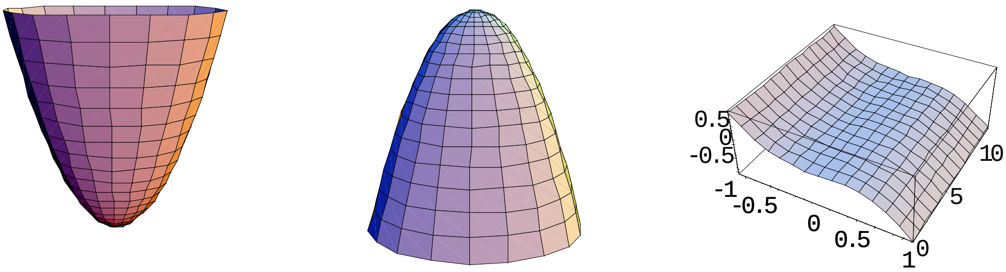

Figure 4

Part 2

3. Tangent plane

3.1. Definition of a surface

Let f be a differentiable function defined on an open set U of □. Let k be a number. The set of points X such that :

$$F(X)=k \quad with \quad gradF(X)\neq 0$$

Is called **surface.**

EXAMPLE

$$F(x,y,z)=x^2+y^2+z^2$$

$F(x,y,z)=1$ is a surface called sphere.

3.2. Proprieties

> Let S be a surface defined on U by:
>
> $F(X)=k$ with $gradF(X)\neq 0$.
>
> The vector **gradf(X)**, perpendicular to all the curves of the surface passing by F(X), is said **normal vector** to the surface S.

In fact:

Let C(t) be a curve for the surface S,

$$F(C(t))=k$$

And let P=C(□) a point of this curve. Let's derivate over t:

$$F'(C(t))=gradF(C(t)).C'(t)=k'\Rightarrow gradF(P).C'(t_0)=0$$

Since C'(□) is the direction of the vector tangent to the curve C(t) at point P we can conclude that in this point the vector gradF(P) is normal to the curve C(t). It's the same for all the curves passing through P and being part of the surface S. GradF(P) is therefore normal to the surface at this point.

EXAMPLE 1

Let S be a surface defined by:

$$F(x,y,z)=x^2+y^2+z^2-1=0$$

And let $P\left(\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2}\right)$ be a point of this surface. Let's consider the 2 following curves

- C(x) is the intersection of surface S with the plane y = x.
    - Write the equation of this curve

- Verify that P is a point of C.
- Find the direction of the tangent to this curve at point P.

- Same question for $\gamma(x)$ intersection of surface S with the plane y =1- x.

SOLUTION

$$C(x) = (x, x, \sqrt{1-2x^2}) \text{ and } \gamma(x) = (x, 1-x, \sqrt{-2x^2+2x})$$

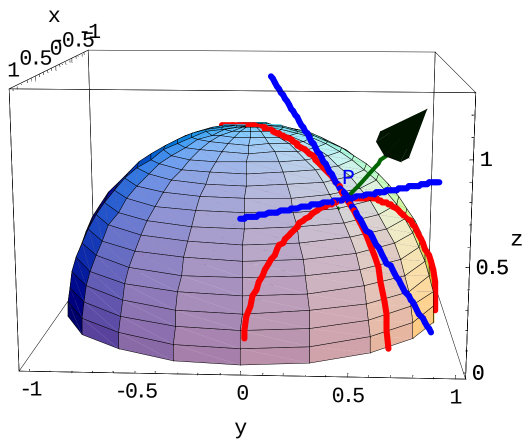

Figure 5

$C'(\frac{1}{2})$ represents the direction of the tangent to the curve C and subsequently to the surface S at point $P\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$

Or: $C'(x) = \left(1, 1, \frac{-2x}{\sqrt{1-2x^2}}\right) \Rightarrow C'\left(\frac{1}{2}\right) = \left(1, 1, -\sqrt{2}\right)$

Similarly $\gamma'(\frac{1}{2})$ represents the tangent to the curve γ and subsequently to the surface S at point P.

$$\gamma'(x) = \left(1, -1, \frac{-2x+1}{\sqrt{-2x^2+2x}}\right) \Rightarrow \gamma'\left(\frac{1}{2}\right) = (1, -1, 0)$$

The normal vector to these 2 tangents is given by:

$$N(\frac{1}{2}) = C'(\frac{1}{2}) \wedge \gamma'(\frac{1}{2}) = (\sqrt{2}, \sqrt{2}, 2)$$

Or

$$gradF(X) = 2(x, y, z) \Rightarrow gradF(\frac{1}{2}) = (1, 1, \sqrt{2}) \Rightarrow gradF(\frac{1}{2}) = \frac{\sqrt{2}}{2} N(\frac{1}{2})$$

This proves that vector $gradF(\frac{1}{2})$ and vector $N(\frac{1}{2})$ are collinear. We can obviously see that vector $gradF(\frac{1}{2})$ is normal to the surface at point P.

EXAMPLE 2

Write the equation of the tangent to the curve $x^2 y + y^3 = 10$ at point P=(1, 2) and define the normal vector to this curve in this point.

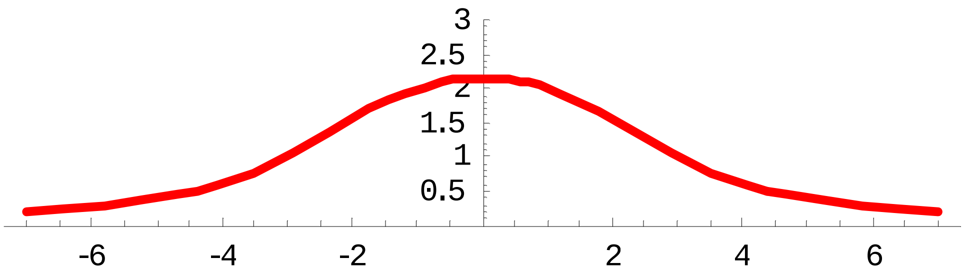

Figure 6

Let's take $f(x,y) = x^2 y + y^3 - 10$.

A normal to this curve is defined by:

$$gradf(x,y) = (2xy, x^2 + 3y^2)$$

At point P: $gradf(P) = (4,13)$

Let X be a point of the tangent. Since P is also a point from this tangent, the direction vector of the tangent is X-P. S o the scalar product between this vector and the one that is normal to the curve is null.

The equation of the tangent to the curve is therefore given by:

$$X.N = N.P \Rightarrow (x,y).(4,13) = (4,13).(1,2) \Rightarrow 4x + 13y = 30$$

3.3. Definition of a tangent plane

Let S be a surface defined on U by: F (X) = k with gradF(X) $\neq 0$. The plane, tangent to the surface S at a point P, is by definition the plane passing through P and having gradF (P) as a normal vector.

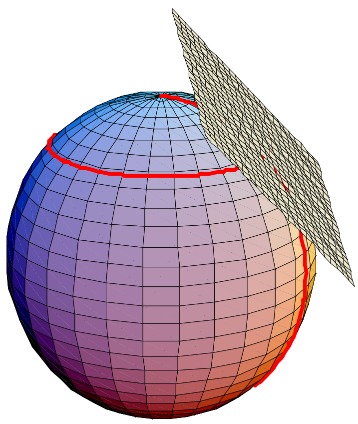

Figure 7

EXAMPLE 3

- Write the equation of the plane that is tangent to the surface: $z = x^2 + y^2$ at point P= (1, 2, 5)

SOLUTION

Let's take $f(x, y, z) = x^2 + y^2 - z$. The normal to this surface is given by:

$$gradf(x, y, z) = (2x, 2y, -1) \Rightarrow gradf(P) = (2, 4, -1)$$

The equation of the plane, tangent to the surface at point P, is given by:

$$X.N = N.P \Rightarrow (x, y, z).(2, 4, -1) = (2, 4, -1).(1, 2, 5)$$

$$\Rightarrow 2x + 4y - z = 5$$

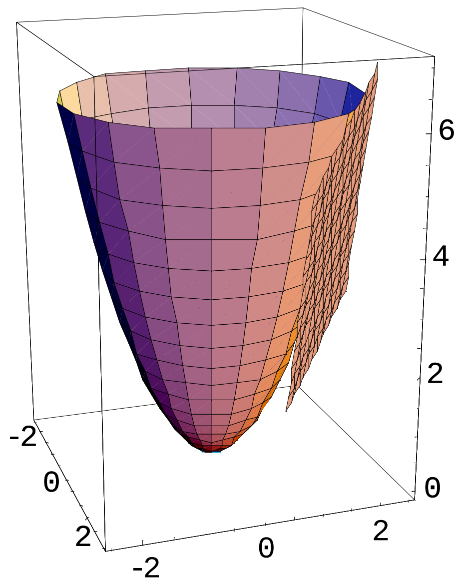

Figure 8

EXAMPLE 4

- Find the parametric equation of the tangent to the curve generated by the intersection of the following two surfaces:

$S_1 : x^2 + y^2 + z^2 = 6$ et $S_2 : x^3 - y^2 + z = 2$ au point P=(1,1,2)

SOLUTION

Let's take $f(x, y, z) = x^2 + y^2 + z^2 - 6$ et $g(x, y, z) = x^3 - y^2 + z - 2$

$$n_1 = gradf(x, y, z) = (2x, 2y, 2z) \Rightarrow gradf(P) = (2, 2, 4) = 2(1, 1, 2)$$

$$n_2 = gradg(x, y, z) = (3x^2, -2y, 1) \Rightarrow gradf(P) = (3, -2, 1)$$

Method 1

The direction vector of the line tangent to these two curve is given by:

$$u = n_1 \wedge n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 3 & -2 & 1 \end{vmatrix} = (1,1,-1)$$

So, the tangent equation would be:

$$M(t) = P + tu \Rightarrow M(t) = (1,1,2) + t(1,1,-1) = (1+t, 1+t, 2-t)$$

Method 2

Equation of the plane, tangent to surface S_1 at point P:

$$(x,y,z).(1,1,2) = (1,1,2).(1,1,2) \Rightarrow x + y + 2z = 6$$

Equation of the plane, tangent to surface S_2 at point P:

$$(x,y,z).(3,-2,1) = (3,-2,1).(1,1,2) \Rightarrow 3x - 2y + z = 3$$

The intersection of these 2 planes is the tangent to surfaces, S_1 and S_2, and passing through P.

$$S_1 \cap S_2 \Rightarrow x + y + 2z = 6x - 4y + 2z \Rightarrow x - y = 0 \Rightarrow y = x$$

$S_1 \cap S_2 \Rightarrow x + z = 3$ in the plane x= y

$$S_1 \cap S_2 \Rightarrow X(t) = (t, t, 3-t)$$

Remark

$$M(-1) = (0,0,3) \Rightarrow M(t) = (0,0,3) + t(1,1,-1) = (t, t, 3-t) \Rightarrow M(t) = X(t)$$

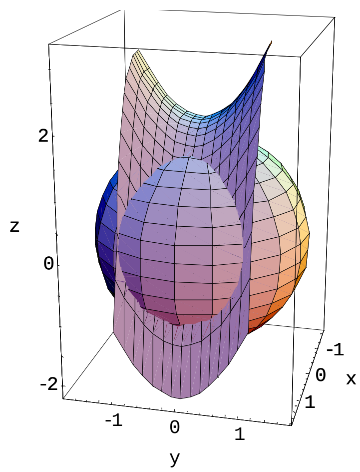

Figure 9

4. Directional derivative

4.1. Definition

Let f be defined on an open set $A \subset R^n$ and assume that f is differentiable. Let P be a point of the open set and u a unit vector. We call directional derivative of f in the direction of u, at point P, the limit, when t=0 of the following:

$$\boxed{D_u f(P) = \lim_{t\to 0} \frac{f(P+tu) - f(P)}{t}}$$

The notation $f_u^{'}(P)$ can also be used.

Since f is differentiable at point P, we have :

$$f(P+tu) - f(P) = gradf(P).tu + \|tu\| g(tu)$$

$$\lim_{t\to 0} \frac{f(P+tu) - f(P)}{t} = \lim_{t\to 0} \frac{gradf(P).tu + \|tu\| g(tu)}{t} = gradf(P).u$$

So $\boxed{D_u f(P) = gradf(P) \cdot u}$

Geometrical interpretation

Let f be defined on an open set $A \subset R^2$ and assume that f is differentiable. Let P(u,v) be a point of the open set and u(a,b) a unit vector. The line passing through P and having u as a direction vector have the following equation: X(t)=P+t u. $f(P+tu)$ is the intersection curve between the surface and the plane P+tu. The direction of the tangent to this curve at point f(P) is given by $f_t^{'}(P)$.

But

$$f_t^{'}(P+tu) = gradf(P+tu) \cdot (P+tu)_t^{'} = gradf(P+tu) \cdot u$$

For t = 0 :

$$f_t^{'}(P) = gradf(P) \cdot u$$

The direction of this tangent is merged with the directional derivative of f at point P in the direction of u .

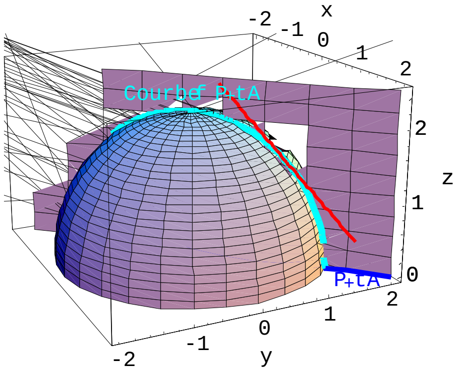

Figure 10

1. **EXAMPLE 1**

Let $f(x,y) = x^2 + y^3$ and v=(1, 2). Compute the directional derivative of f in the direction of v at point P=(-1,3)

Since v is not a unit vector, lets take vector $u = \frac{v}{\|v\|} = \frac{1}{\sqrt{5}}(1,2)$

$$gradf(x,y) = (2x, 3y^2) \Rightarrow gradf(P) = (-2,27)$$

$$\Rightarrow D_u f(P) = gradf(P.u) = (-2,27).\frac{1}{\sqrt{5}}(1,2) = \frac{52}{\sqrt{5}}$$

The tangent direction to surface $z = x^2 + y^3$ at point (P, f(P)) is therefore given by:

$$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{52}{\sqrt{5}}\right)$$

4.2. Maximum of the directional derivative

We that for a unit vector u :

$$D_u f(P) = gradf(P).u$$

Let's note θ the angle between $gradf(P)$ and vector u. So :

$$D_u f(P) = \|gradf(P)\|.\|u\|\cos(\theta) = \|gradf(P)\|\cos(\theta)$$

Since the maximum of $\cos(\theta)$ is 1 and since $\|gradf(P)\|$ is a positive constant, the maximum of $D_u f(P)$ is reached when $\cos(\theta) = 1$ that means when the unit vector u is in the direction of $gradf(P)$.

2. EXAMPLE 2

In the preceding example u should be:

$$u = \frac{gradf(P)}{\|gradf(P)\|} = \frac{1}{\sqrt{733}}(-2,27)$$

The maximum of $D_u f(P)$ is therefore $\|gradf(P)\| = \sqrt{733}$

3. EXAMPLE 3

Write the equation of the tangent to the following curve in the direction A at point $(x_P, y_P, f(P))$:

$$U = R^2;\ f(x,y) = x^2 + 2y;\ A = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right);\ P = (2,4)$$

$$\Rightarrow (x_P, y_P, f(P)) = (2,4,12)$$

The direction of this tangent is:

Method 1

$$\Rightarrow P + tA = \left(2 + \frac{t}{\sqrt{5}}, 4 + \frac{2t}{\sqrt{5}}\right) \Rightarrow f(P + tA) = \left(2 + \frac{t}{\sqrt{5}}\right)^2 + 2\left(4 + \frac{2t}{\sqrt{5}}\right)$$

$$\Rightarrow f(P+tA)=\frac{t^2}{5}+\frac{8}{\sqrt{5}}t+12 \Rightarrow f_t^{'}(P+tA)=\frac{2}{5}t+\frac{8}{\sqrt{5}} \Rightarrow f_t^{'}(P)=\frac{8}{\sqrt{5}}$$

Method 2

$$gradf(X)=(2x,2)\Rightarrow gradf(P)=(4,2)\Rightarrow D_A f(P)=gradf(P).A=\frac{4}{\sqrt{5}}+\frac{4}{\sqrt{5}}=\frac{8}{\sqrt{5}}$$

Tangent equation

The equation of the tangent to the surface at point (x_P, y_P, z_P) in the direction A, is the equation of the line passing through point (x_P, y_P, z_P) and having this direction vector:

$$u=\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}},\frac{8}{\sqrt{5}}\right)$$

In other terms

$$X(t)=(x_P, y_P, z_P)+tu\Rightarrow X(t)=(2,4,12)+t\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}},\frac{8}{\sqrt{5}}\right)$$

$$\Rightarrow X(t)=\left(2+\frac{t}{\sqrt{5}},4+\frac{2t}{\sqrt{5}},12+\frac{8t}{\sqrt{5}}\right)$$

4. EXAMPLE 4

$$U=R^2;\ f(x,y)=x^2+y^2;\ A=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right);\ P=(1,0)\ \Rightarrow P+tA=\left(1-\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}}\right)$$

$$gradf(X)=(2x,2y)\Rightarrow gradf(P)=(2,0)\Rightarrow gradf(P).A=-\sqrt{2}$$

TANGENT

The tangent is the line passing through point (x_P, y_P, z_P) and parallel to vector:

$$u=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\sqrt{2}\right)$$

So

$$X=(x_P, y_P, z_P)+tu\Rightarrow X=(1,0,1)+t\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},-\sqrt{2}\right)\Rightarrow$$

$$X=\left(1-\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}},1-\sqrt{2}t\right)$$

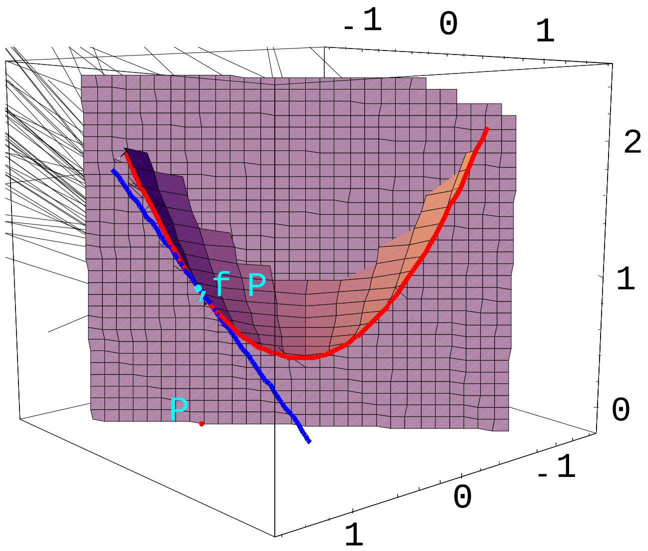

Figure 11

5. EXAMPLE 5

Let $i = e_1 = (1,0)$ and $j = e_2 = (0,1)$. Let P=(a,b). Compute $D_i f(P)$ et $D_j f(P)$

$$D_i f(P) = gradf(P).(1,0) = \left(f_x^{'}(P), f_y^{'}(P)\right).(1,0) = f_x^{'}(P)$$

But

$$f_x^{'}(P) = \lim_{t\to 0}\frac{f(P+ti) - f(P)}{t} = \lim_{t\to 0}\frac{f(a+t,b) - f(a,b)}{t} = D_i f(P)$$

Similarly:

$$D_j f(P) = gradf(P).(0,1) = \left(f_x^{'}(P), f_y^{'}(P)\right).(0,1) = f_y^{'}(P)$$

$$f_y^{'}(P) = \lim_{t\to 0}\frac{f(P+tj) - f(P)}{t} = \lim_{t\to 0}\frac{f(a,b+t) - f(a,b)}{t} = D_j f(P)$$

6. EXAMPLE 6

Let function f : $z = f(x,y) = \sqrt{4 - x^2 - y^2}$ and let point P (1,1) and vector $u = \frac{1}{\sqrt{2}}(1,1)$

Compute $D_u f(P)$

FIRSTE METHODE

$$P + tu = (1,1) + \frac{t}{\sqrt{2}}(1,1) = (1 + \frac{t}{\sqrt{2}}, 1 + \frac{t}{\sqrt{2}})$$

$$gradf((x,y) = (\frac{-x}{\sqrt{4 - x^2 - y^2}}, \frac{-y}{\sqrt{4 - x^2 - y^2}}) \Rightarrow gradf(P) = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$$

$$D_u f(P) = gradf(P)\cdot u = -\frac{1}{\sqrt{2}}(1,1)\cdot\frac{1}{\sqrt{2}}(1,1) = \frac{1}{2}(1+1) = -1$$

SECOND METHODE

The plane passing through the line $P+tu$ and parallel to the z axis have the following equation y=x. The intersection curve between this plane and the hemisphere is given by:

$$C(t) = f(P+tu) = \sqrt{4-2(1+\frac{t}{\sqrt{2}})^2}$$

$$\Rightarrow C'(t) = \frac{-\sqrt{2}(1+\frac{t}{\sqrt{2}})}{\sqrt{4-2(1+\frac{t}{\sqrt{2}})^2}} \Rightarrow C'(0) = D_u f(P) = -\frac{\sqrt{2}}{\sqrt{2}} = -1$$

TANGENT

The equation of the tangent to the surface at point (x_P, y_P, z_P), in the u direction, is the equation of the line passing through point (x_P, y_P, z_P) and parallel to vector u:

$$u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right)$$

That means

$$X = (x_P, y_P, z_P) + tu \Rightarrow X(t) = (1,1,\sqrt{2}) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right)$$

$$\Rightarrow X(t) = (1,1,\sqrt{2}) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right)$$

$$\Rightarrow X(t) = \left(1+\frac{t}{\sqrt{2}}, 1+\frac{t}{\sqrt{2}}, \sqrt{2}-t\right)$$

7. EXAMPLE 7

Let f be the function defined by $f(x,y) = x^2 + y^2$. This is a revolution surface around the z'z axis because $z = f(r)$. Let the point P (1,1) and vector $u = \frac{1}{\sqrt{5}}(2,1)$. Compute $D_u f(P)$

FIRST METHODE

$$P+tu = (1,1) + \frac{t}{\sqrt{5}}(2,1) = (1+\frac{2t}{\sqrt{5}}, 1+\frac{t}{\sqrt{5}}) \ ;$$

$$gradf((x,y) = (2x,2y) \Rightarrow gradf(P) = (2,2)$$

$$D_u f(P) = gradf(P)\cdot u = (2,2)\cdot\frac{1}{\sqrt{5}}(2,1) = \frac{1}{\sqrt{5}}(4+2) = \frac{6}{\sqrt{5}}$$

SECOND METHODE

The plane passing through the line $P+tu$ and the z axis have the following equation y=x. This plane cuts the hemisphere at:

$$C(t)=f(P+tu)=\left(1+\frac{2t}{\sqrt{5}}\right)^2+\left(1+\frac{t}{\sqrt{5}}\right)^2$$

$$\Rightarrow C'(t)=2t+\frac{6}{\sqrt{5}} \Rightarrow C'(0)=D_u f(P)=\frac{6}{\sqrt{5}}$$

TANGENT

Let $z_P = f(P)$

$$u=\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},\frac{6}{\sqrt{5}}\right) \text{ c.à d. } X(t)=\left(x_P,y_P,z_P\right)+tu$$

$$\Rightarrow X(t)=(1,1,2)+t\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},\frac{6}{\sqrt{5}}\right) \Rightarrow X(t)=\left(1+\frac{2t}{\sqrt{5}},1+\frac{t}{\sqrt{5}},2+\frac{6t}{\sqrt{5}}\right)$$

5. Additional derivation techniques

Let's consider the case of a 2 variables, u and v, function where both u and v depend of 2 variables x and y. u= u(x, y) and v= v(x, y)

$$f(x,y)=F[u(x,y),v(x,y)]$$

By applying the derivation rules (taking x than y as constants) we get:

$$\frac{\partial f}{\partial x}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial f}{\partial y}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}$$

Here's the matrix presentation of this result:

$$\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial F}{\partial u}\\ \frac{\partial F}{\partial v}\end{pmatrix}$$

where the matrix:

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}u'_x & v'_x\\ u'_y & v'_y\end{pmatrix}$$

is said "**Jacobeans matrix**" of u and v with respect to x and y.