Chapter IV

Potential Functions

1. Differentiable curve

1.1. Definitions

Case of 2 variables

Let U be an open set of R^2. We shall name a differentiable form on U every w such that there exist 2 applications $P, Q : U \to R$, of **class** C^1 over U, that verify the following:

$$\forall (x, y) \in U, \quad w(x, y) = P(x, y)dx + Q(x, y)dy$$

Case of 3 variables

Let U be an open set of R^3. We shall name a differentiable form on U every w such that there exist 3 applications $P, Q, R : U \to R$, of **class** C^1 over U, that verify the following:

$$\forall (x, y, z) \in U, \quad w(x, y, z) = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

1.2. Exact differentiable forms

Definition

Let U be an open set of R^2 and w a differentiable form over U. we shall say that w is exact over **U** (or: w have an integral over U) if and only if there exist a function $f : U \to R$ of class C^1 over U such that :

$$\forall (x, y) \in U, \quad df(x, y) = w(x, y)$$

Using P and Q:

$$\forall (x, y) \in U, \quad w(x, y) = P(x, y)dx + Q(x, y)dy$$

The relation $df(x, y) = w(x, y)$ becomes:

$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = P(x, y) \\ \dfrac{\partial f}{\partial y}(x, y) = Q(x, y) \end{cases} \quad \text{where} \quad \begin{cases} f_x^{'}(x, y) = P(x, y) \\ f_y^{'}(x, y) = Q(x, y) \end{cases}$$

Same definition applies for a 3 variables function

Examples

$$w(x, y) = xdx + ydy = d\left(\frac{1}{2}(x^2 + y^2)\right)$$

$$w(x,y) = xdy + ydx = d(xy)$$

$$w(x,y) = \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy = d\left(\frac{1}{2}\ln(x^2+y^2)\right)$$

1.3. Closed differentiable forms

Definition

Case of 2 variables

Let U be an open set of R^2 and w a differentiable form over U, defined by:

$$\forall (x,y)\in U,\quad w(x,y) = P(x,y)dx + Q(x,y)dy$$

We shall say that w is closed over U if and only if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{or} \qquad P'_y(x,y) = Q'_x(x,y)$$

Case of 3 variables

Let U be an open set of R^3 and w a differentiable function over U, defined by:

$$\forall (x,y,z)\in U,\quad w(x,y,z) = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

We shall say that w is closed over U if and only if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},\ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad \text{or} \qquad P'_y = Q'_x,\ Q'_z = R'_y,\ R'_x = P'_z$$

Hint, to memorize quickly:

Let's define the **rotational** vector by:

$$rot(P,Q,R) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

A differentiable form w is closed if $rot(P,Q,R) = 0$

Notation

If the form is written:

$$w(x,y,z) = f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz$$

The differentiable form w is closed if:

$$rot\left(f_1, f_2, f_3\right) = 0 \Leftrightarrow \forall i \neq j, \quad D_i f_j = D_j f_i$$

Theorem

> Every exact differentiable form is closed.

We shall demonstrate this theorem in R^2

DEMONSTRATION

Let $w(x,y) = P(x,y)dx + Q(x,y)dy$ an exact differentiable form over an open set U. This is equivalent to saying that there exists a function f of class C^1 such that:

$$df(x,y) = w(x,y).$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x}(x,y) = P(x,y) \\ \dfrac{\partial f}{\partial y}(x,y) = Q(x,y) \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 f}{\partial y \partial x}(x,y) = \dfrac{\partial P}{\partial y}(x,y) \\ \dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial Q}{\partial x}(x,y) \end{cases}$$

P and Q being of class C^1 over U $\Rightarrow \dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$ are continuous

$\Rightarrow \dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ are continuous + Schwartz theorem:

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) \Rightarrow \frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$$

Definition

Let $\boldsymbol{X}$ be a closed set of R^2 or R^3

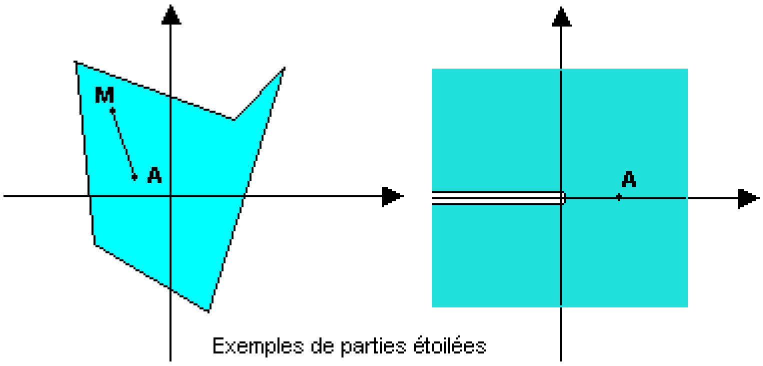
1) Let $A \in X$; we say that $\boldsymbol{X}$ **is starred with respect to** $\boldsymbol{A}$ if and only if:

$$\forall M \in X, \quad [AM] \subset X$$

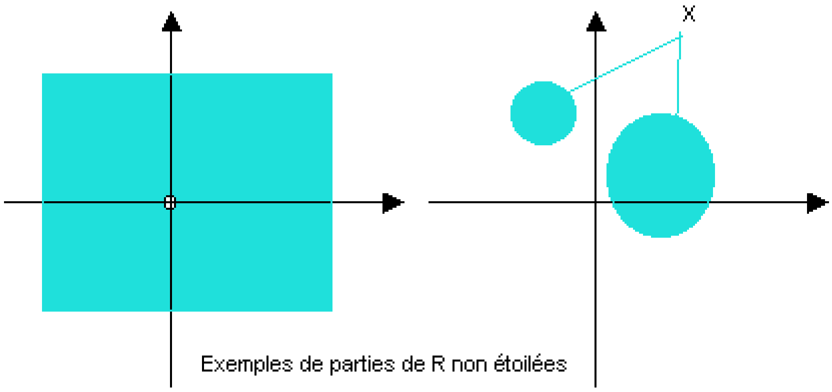
where $[AM]$ is the segment joining A and M, which means

$$[AM] = \{P \in R^2 \text{ ou } R^3; \exists \lambda \in [0,1], \left(P - A\right) = \lambda\left(M - A\right)\}$$

2) We shall say that $\boldsymbol{X}$ is **starred** if and only if there exist a point $A \in X$ such that $\boldsymbol{X}$ **is starred with respect to** $\boldsymbol{A}$.



Exemples de parties étoilées



Exemples de parties de R non étoilées

Poincare Theorem

> Let U be a **starred** open set of R^3 (or R^2) and w a differentiable form over U. If w **is closed** over U then w is **exact** over U.

Example

Study the differentiable form w defined over R^2 by:

$$w(x,y) = (3x^2y + 2x + y^3)dx + (x^3 + 3xy^2 - 2y)dy$$

Let's put: $P(x,y) = 3x^2y + 2x + y^3$ et $Q(x,y) = x^3 + 3xy^2 - 2y$

We have: $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} = 3\left(x^2 + y^2\right)$

The differentiable form w is therefore closed. Since R^2 is starred ➔ w is exact.

This can verified using the following function:

$$f(x,y) = \left(x^2 - y^2\right) + xy\left(x^2 + y^2\right)$$

where: $df(x,y) = f_x^{'}(x,y)dx + f_y^{'}(x,y)dy = w(x,y)$

2. Vector field

Definition

Let U be an open set. By a vector filed on U we mean an association which to every point of U associates a vector of the same dimension.

$$F : R^n \to R^n$$

EXAMPLE

- Let $F(x, y) = (x^2 y, \sin xy)$. It's a vector filed defined over R^2.

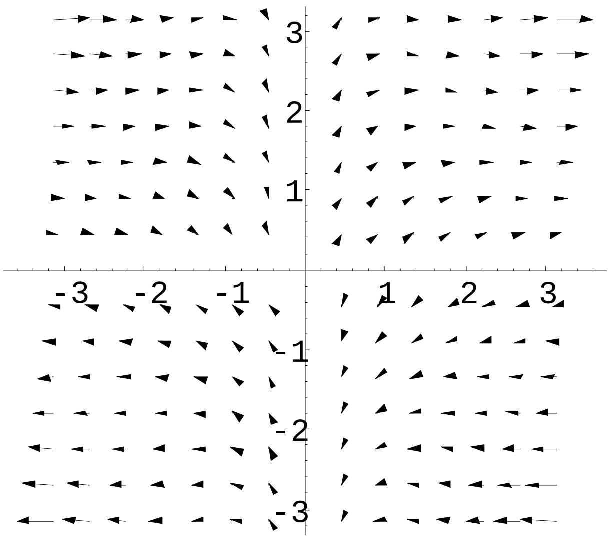


Figure 1

- The vector field $gradf(x, y) = \left(f_x^{'}(x, y),\ f_y^{'}(x, y)\right)$ is a vector field associated to the differentiable function f . It is called **gradient field.**

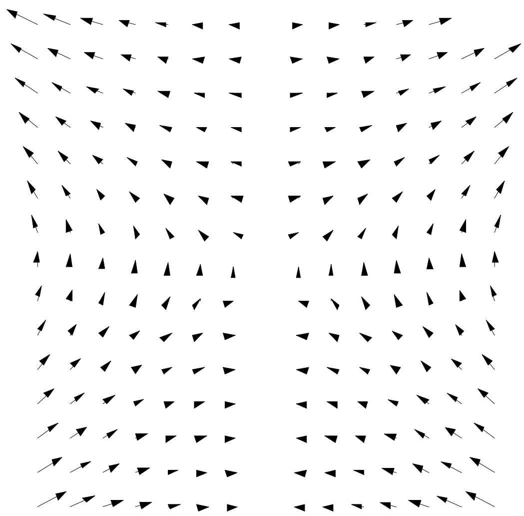


Figure 2

3. Potential functions

3.1. Definition

Let F a vector field defined over an open set U.

Let φ a differentiable function defined over U and verifying F = gradφ. φ is called **potential function** of F.

If F = - gradφ , φ is called **potential energy** of F.

3.2. Existence of a potential function

The existence of a potential function relative to a vector field:

$$F(x,y)=(P(x,y),Q(x,y))$$

is a problem identical to finding if the form $P(x,y)dx+Q(x,y)dy$ is exact.

EXAMPLE 1

Let $F(x, y)=(x^2y,\sin xy)$. Let's consider $P(x, y)=x^2y$ and $Q(x, y)=\sin xy$.

So: $\frac{\partial P}{\partial y}=x^2;\ \frac{\partial Q}{\partial x}=y\cos xy \ \Rightarrow \ \frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$

➔ F does not have a potential function.

EXAMPLE 2

Let $F(x,y)=(e^{xy},e^{x+y})$, does it have a potential function?

Let's consider $f(x,y)=e^{xy}$ et $g(x,y)=e^{x+y}$

$\Rightarrow \ \frac{\partial f}{\partial y}=xe^{xy};\quad \frac{\partial g}{\partial x}=e^{x+y}\Rightarrow\frac{\partial f}{\partial y}\neq\frac{\partial g}{\partial x}$

➔ F does not have a potential function.

EXAMPLE 3

Let $F(x,y)=(2xy,x^2+y^2)$ does it have a potential function?

Let's consider f(x, y)=2xy and $g(x,y)=x^2+y^2$

$\Rightarrow \ \frac{\partial f}{\partial y}=2x;\ \frac{\partial g}{\partial x}=2x\Rightarrow\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$

Since F is defined over $\mathbb{R}^2$ and $\mathbb{R}^2$ is starred, so it has a potential function. Let φ be this function. φ verifies:

$$\frac{\partial\varphi}{\partial x} = 2xy \text{ and } \frac{\partial\varphi}{\partial y} = x^2 + y^2$$

$$\varphi(x,y) = \int 2xy\,dx + \alpha(y) = x^2 y + \alpha(y)$$

$$\Rightarrow \frac{\partial\varphi}{\partial y} = x^2 + \alpha'(y) \Rightarrow \alpha'(y) = y^2 \Rightarrow \alpha(y) = \frac{y^3}{3} + C$$

Finally we get: $\boxed{\varphi(x,y) = x^2 y + \frac{y^3}{3} + C}$

$$\Rightarrow (2xy, x^2 + y^2) = grad(x^2 y + \frac{y^3}{3} + C)$$

EXAMPLE 4

Let $F(x,y,z) = (y\cos xy, x\cos xy + 2yz^3,\ 3y^2z^2)$. Does it have a potential function f?

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = y\cos xy \\ \frac{\partial f}{\partial y}(x,y,z) = x\cos xy + 2yz^3 \\ \frac{\partial f}{\partial z}(x,y,z) = 3y^2z^2 \end{cases}$$

$$\Rightarrow \frac{\partial f}{\partial z}(x,y,z) = 3y^2z^2 \Rightarrow f(x,y,z) = y^2z^3 + \alpha(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial x}(x,y,z) = \alpha'_x(x,y) = y\cos xy \Rightarrow \alpha(x,y) = \int y\cos xy\,dx = \sin xy + \beta(y)$$

$$\Rightarrow f(x,y,z) = y^2z^3 + \sin xy + \beta(y) \Rightarrow \frac{\partial f}{\partial y}(x,y,z) = 2yz^3 + x\cos xy + \beta'(y)$$

$$\Rightarrow \beta'(y) = 0 \Rightarrow \beta(y) = C \Rightarrow f(x,y,z) = y^2z^3 + \sin xy + C$$

$$\Rightarrow (y\cos xy, x\cos xy + 2yz^3, 3y^2z^2) = grad(y^2z^3 + \sin xy + C)$$

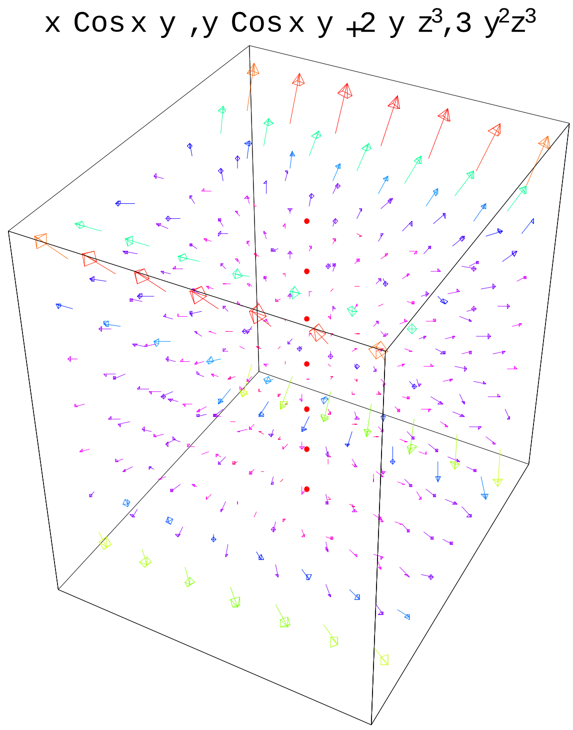


Figure 3

EXAMPLE 5

Let $G(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$ does it have a potential function?

Let's consider $f(x,y) = \dfrac{-y}{x^2+y^2}$ and $g(x,y) = \dfrac{x}{x^2+y^2} \Rightarrow \dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x} = \dfrac{-x^2+y^2}{\left(x^2+y^2\right)^2}$

But this function does not have a potential function in R^2 because it's not continuous at (0,0).

Oppositely, if we take $\varphi(x,y) = Arctg(\dfrac{y}{x}) + c$ with $x \neq 0 \Rightarrow G(x,y) = grad\,\varphi(x,y)$

G has a potential function in U, starred and not crossing the y axis.

If we use the polar coordinates we get:

$$\varphi(x,y) = Arctg(\frac{y}{x}) \Rightarrow \varphi^*(r,\theta) = \theta$$

So: $\dfrac{-y}{x^2+y^2}dx + \dfrac{x}{x^2+y^2}dy = d\theta$

This result will be used later on.

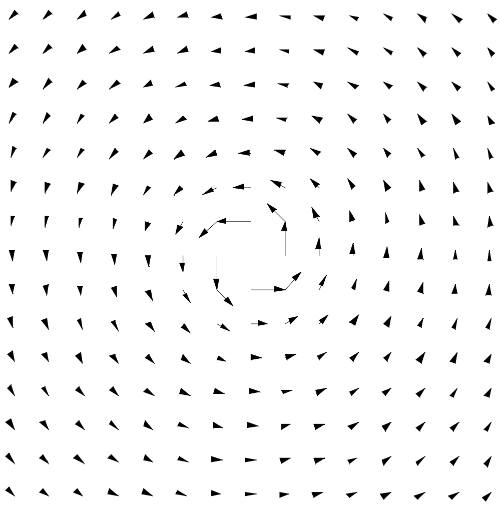


Figure 4

EXAMPLE 6

Let $F(x,y)=\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)$, does it have a potential function?

Let $P=\frac{x}{x^2+y^2}$ et $Q=\frac{y}{x^2+y^2}$ => $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=-\frac{2xy}{\left(x^2+y^2\right)^2}$

P and Q are defined over $R^2-\{0,0\}$ which is not starred. Let's consider a starred domain U of R^2 that does not contain the point $\{0,0\}$. In this domain it is easy to prove that:

$$F(x,y)=grad\left(\frac{1}{2}Log\left(x^2+y^2\right)\right)=grad\,\varphi(x,y)$$

$\varphi(x,y)$ is defined over $R^2-\{0,0\}\Rightarrow$ The vector field $F(x,y)$ have a potential function inside $R^2-\{0,0\}$

4. Derivation inside the integral sign

4.1. Theorem

Let's consider that a 2 variables function f and it's partial derivative D_2f exist and are continuous in the rectangle [a, b]×[c, d].

Let $F(y)=\int_a^b f(x,y)dx$

So, F is derivable with respect to y and

$$F'(y)=\int_a^b f_y'(x,y)\,dx \quad \text{where} \quad \frac{\partial F}{\partial y}(x,y)=\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

4.2. Generalization: Formula of Leibniz

$$F(y)=\int_{u(y)}^{v(y)} f(x,y)dx$$

$$\Rightarrow F'(y)=\int_{u(y)}^{v(y)} f_y'(x,y)\,dx + f(v(y),y)v_y' - f(u(y),y)u_y'$$

EXAMPLES

- $f(y)=\int_y^{y^2} x^2(y-x)^3\,dx$
  - $f'(y)=\int_y^{y^2} 3x^2(y-x)^2\,dx+2y^5\left(y-y^2\right)^3$
- $g(x,y)=\int_1^x e^{-t^2-y^2}\,dt$
  - $g_x'(x,y)=e^{-x^2-y^2}$
  - $g_y'(x,y)=\int_1^x -2ye^{-t^2-y^2}\,dt$