Chapter V

Curve integrals

1. Definition and evaluation of curve integrals

In all the following, we will assume that n=2 or n=3 (the most important cases) and that every field vector F and curve C are at least of class C^1. Much of what we say will be true in general (unspecified number of variables)

1.1 Introduction

Let F be a vector field on an open set U in the plane. We interpret F as a field of forces such as the wind. A plane flies over a curve $C(t)$ from point $C(t_0)$ to point $C(t_1)$.

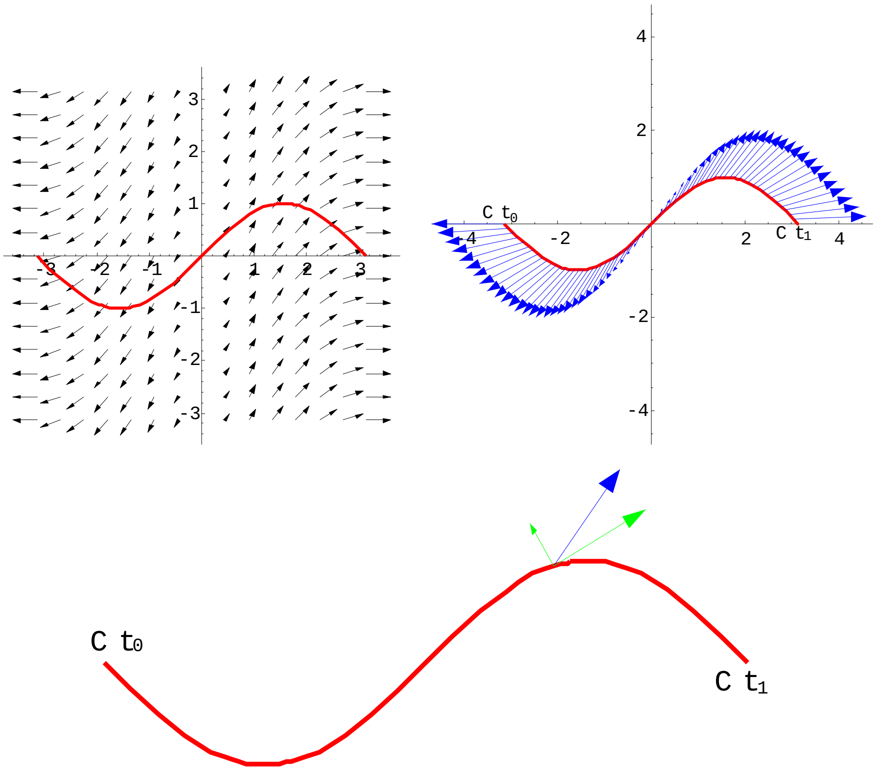


Figure 1

To find the work done against this force field along the curve we shall first take the component of the force along the curve. This is given by a dot product F(C(t)).C'(t), then we integrate this function along the curve and interpret the result as the work.

Definition

Let U be an open set and F a vector field defined in U. Let C(t) a curve in U defined over [a,b]. We shall call curve integral of F along C $\int_C F$ the following integral:

$$\int_C F = \int_a^b F(C(t)).C'(t)dt$$

The scalar product F(C(t))•C'(t) is a function of t, it's the projection of F(C(t)) over the tangent C'(t), to the curve C.

REMARK 1

If C(a)=P and C(b)=Q we can also write: $\int_C F = \int_{P,C}^{Q} F = \int_{P,C}^{Q} F(C(t)).dC(t)$

REMARK 2

If F(C(t)) is the unit vector field held by the tangents to the curve C(t),

$$F(C(t)) = \frac{(x'(t), y'(t))}{\sqrt{x'^2(t) + y'^2(t)}}$$

The curve integral is interpreted as the length of the curve C

$$\int_C F = \int_a^b \frac{(x'(t), y'(t))}{\sqrt{x'^2(t) + y'^2(t)}}.(x'(t), y'(t))dt = \int_a^b \sqrt{x'^2(t) + y'^2(t)}dt = L$$

REMARK 3

If the field F and the curve C are given by:

$$F(x,y) = \left(P(x,y), Q(x,y)\right) \; ;$$
$$C(t) = \left(x(t), y(t)\right) \Rightarrow C'(t)dt = \left(x'(t)dt, y'(t)dt\right) = (dx, dy)$$

then:

$$\int_{P,C}^{Q} F = \int_{P,C}^{Q} P(x,y)dx + Q(x,y)dy$$

1.2 Curve integrals, computation techniques

EXAMPLE 1

Compute the curve integral of the following vector field $F(x,y) = (xy, y^2)$ along the segment joining the point O(0,0) and the point P(1,1).

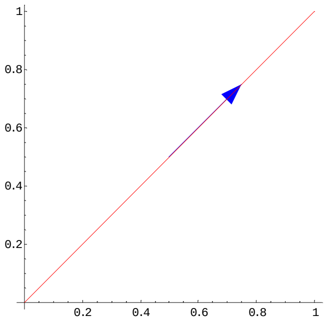


Figure 2

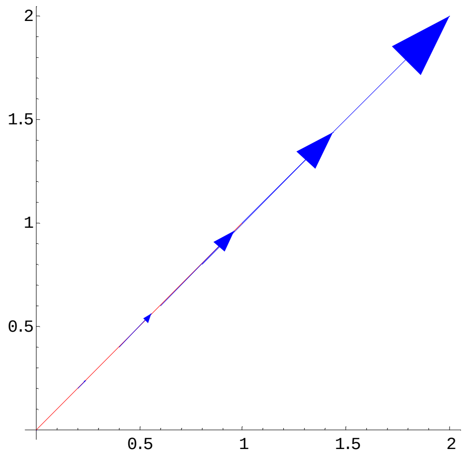


Figure 3

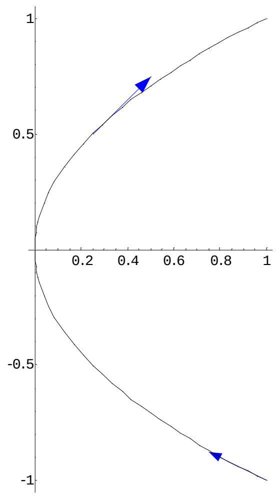
The segment OP has the following parametrical representation:

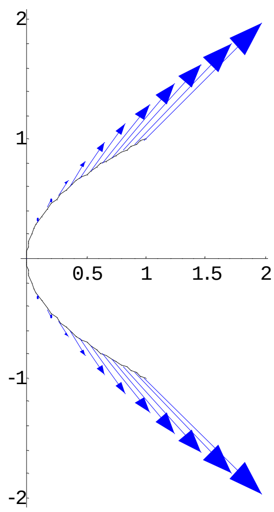
$$C(t)=t\,P \Rightarrow C(t) = (t, t) \Rightarrow C'(t) = (1, 1)$$

$$\Rightarrow \int_C F = \int_0^1 (t^2, t^2).(1,1)dt = \frac{2}{3}$$

EXAMPLE 2

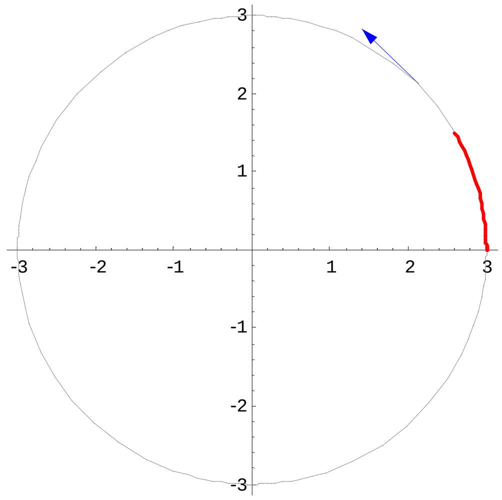
Find the circulation of $F(x, y) = (x^2, xy)$ along the following parabola $x = y^2$ between (1,-1) and (1, 1). $\Rightarrow \int_C F = \int_{-1}^{1} (t^4, t^3).(2t,1)dt = 0$





EXAMPLE 3

Circulation of the field $G(x,y)=\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$ along the circle (O(0, 0), r=3) in the trigonometric direction from point (3, 0) to point $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$.



The curve integral between (3, 0) and $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$ exist because if G(x, y) does not exist at point (0, 0), there surely exist an open set U containing the arc in which G(x, y) is differentiable.

The parametrical equation of the specified arc is:

$$C(\theta)=(3\cos\theta,3\sin\theta)\quad 0\le\theta\le\pi/6$$

$$\Rightarrow C\left(\frac{\pi}{6}\right)=3\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right);\ \text{C(0)=(3, 0)}$$

$$\Rightarrow C'(\theta)=(-3\sin\theta,3\cos\theta)\quad 0\le\theta\le\pi/6$$

$$\Rightarrow G(C(\theta))=\left(\frac{-3\sin\theta}{9},\frac{3\cos\theta}{9}\right)=\frac{1}{3}(-\sin\theta,\cos\theta)$$

$$\Rightarrow \int_C G=\int_0^{\pi/6}\frac{1}{3}(-\sin\theta,\cos\theta)(-3\sin\theta,3\cos\theta)d\theta=\pi/6$$

CONCLUSION

We have re-demonstrated that:

$$\frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy=d\theta$$

1.3 Closed paths

Definition

If the curve is the union of a finite number of arcs $\{C_1, C_2, ..., C_m\}$ where C_i is defined over $[a_i, b_i]$ in such a way that $C_i(b_i) = C_{i+1}(a_{i+1})$, we define:

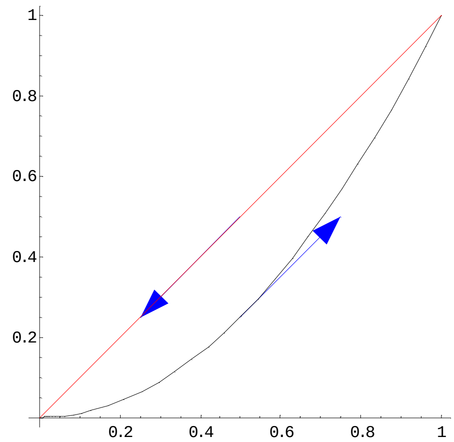
$$\int_C F = \int_{C_1} F + \int_{C_2} F + ... + \int_{C_m} F$$

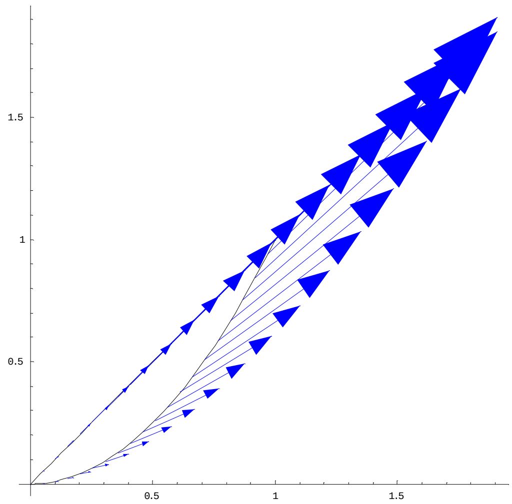
We shall say that C is closed if $C_m(b_m) = C_1(a_1)$

EXAMPLE 4

Let $F(x, y) = (x^2, xy)$.

Compute the circulation of F along the closed curve C made from the line segment joining point (1, 1) to point (0, 0) and the segment of the parabola $y = x^2$ between (0, 0) and (1, 1).





$$\int_C F = \int_{C_1} F + \int_{C_2} F = \frac{1}{15}$$

Computation of $\int_{C_1} F$

$$y = x^2 \Rightarrow C(t) = (t, t^2); \quad 0 \le t \le 1 \Rightarrow C'(t) = (1, 2t)$$

$$F(x, y) = (x^2, xy) \Rightarrow F(C(t)) = (t^2, t^3)$$

$$F(C(t)).C'(t) = (t^2, t^3).(1, 2t) = t^2 + 2t^4$$

$$\Rightarrow \int_{C_1} F = \int_0^1 (t^2 + 2t^4)\, dt = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}$$

Computation of $\int_{C_2} F$

$$y = x \Rightarrow C(t) = (t, t); \quad \text{t:}1 \to 0 \Rightarrow C'(t) = (1, 1)$$

$$F(x, y) = (x^2, xy) \Rightarrow F(C(t)) = (t^2, t^2)$$

$$F\left(C\left(t\right)\right).C'\left(t\right)=\left(t^2,t^2\right).(1,1)=2t^2$$

$$\int_{C_2} F=\int_1^0 2t^2dt=-\frac{2}{3}\Rightarrow \int_C F=\int_{C_1} F+\int_{C_2} F=\frac{11}{15}-\frac{2}{3}=\frac{1}{15}$$

1.4 Reverse path

If in the preceding example we have reversed the direction along the curves, we would have obtained $-\frac{1}{15}$. We shall admit the following notations:

If C is the curve going from P to Q, C^- would be the same curve, but going from Q to P.

If C(t) is the curve defined over [a, b], we define over [a, b] the curve $C^-(t)$ by:

$C^-(t)=C(a+b-t)$

In fact:

$C^-(a)=C(b);\ C^-(b)=C(a)$

and when t increase from a to b, a+b-t decrease from b to a. Therefore, C^- is nothing but C, but travelled in the opposite direction.

Lemma

Let U be an open set and F a vector field defined in U. Let C be a curve defined over [a,b]. Then: $\int_{C^-} F=-\int_C F$

IN FACT

$$\int_{C^-} F=\int_b^a F(C^-(t))\cdot C^-{}'(t)dt=\int_b^a F(C(a+b-t)).C'(a+b-t)dt$$

$$=\int_a^b F(C(u)).C'(u)(-du)=-\int_C F$$

2. Curve integrals when the vector field has a potential fucntion

When the vector field F admits a potential function f, the integral of F along a curve has a simple expression in terms of f.

Théorème 1

Let F be a vector field on the open set U and assume that admits a potential function f on U (F=grad f). Let C be a curve joining the points P and Q. Then:

$$\int_{P,C}^{Q} F=\varphi(Q)-\varphi(P)$$

In particular, the integral of F is independent of the curve joining P and Q.

IN FACT

$$\int_{P,C}^{Q} F = \int_a^b F(C(t)).C'(t)dt = \int_a^b grad\,\varphi(C(t)).C'(t)dt$$

$$= \int_a^b \varphi_t'(C(t))dt = \varphi(C(t))\Big]_a^b = \varphi(Q) - \varphi(P)$$

Corollary

Let F be a vector field on an open set U. If F has a potential function, then the integral of F along every closed path in U is equal to 0. If there exists a closed path C in U such that:

$$\int_\Gamma F \neq 0$$

Then F does not have a potential function.

EXAMPLE 5

Let $F(x,y,z) = (2xy^3z, 3x^2y^2z, x^2y^3)$

here $F(x,y,z) = grad(x^2y^3z)$,

so the integral of F over a path joining between P(1, -1, 2) and Q(-3, 2, 5) is:

$$\int_P^Q F = \varphi(Q) - \varphi(P) = \left[x^2y^3z\right]_P^Q = \left((-3)^2\, 2^3 5 - (-1)^3\, 2\right) = 362$$

Theorem 2

Let U be a connected open set and let F be a vector field on U. Assume that given 2 points P and Q in U, the integral $\int_{P,C}^{Q} F$ is independent of the path C joining P and Q. Then there exist a potential function for F on U.

PROOF

We select some fixed point P_0 in U and for an arbitrary point X in U, we define:

$$\varphi(X) = \int_P^X F$$

This integral is independent of the path joining P and Q, so the path is not mentioned next to integral sign. Let's assume $F = (f_1, f_2, ..., f_n)$ and $E_i = (0,0,...,\overset{(i)}{1},...,0)$ so:

$$F(X).E_i = f_i(X)$$

Let's show that F=gradφ i.e. $\forall i \;\; f_i(X) = D_i\varphi(X)$

Therefore let's consider the Newton quotient:

$$\frac{\varphi(X+hE_i)-\varphi(X)}{h}=\frac{1}{h}\left[\int_P^{X+hE_i}F-\int_P^X F\right]$$
$$=\frac{1}{h}\left[\int_{P_0}^{X}F+\int_X^{X+hE_i}F-\int_{P_0}^X F\right]$$
$$=\frac{1}{h}\int_X^{X+hE_i}F$$

In fact we take C to be the parameterized straight line segment between X and X+hE_i given by:

C(t) = X + t (X + hE_i - X)=X + t hE_i $\Rightarrow$ C'(t) = h E_i

But

$$\int_{X,C}^{X+hE_i}F=\int_X^{X+hE_i}F(C(t)).C'(t)dt$$

so

$$\frac{\varphi(X+hE_i)-\varphi(X)}{h}=\frac{1}{h}\left[\int_0^1 F(C(t)).hE_i dt\right]$$
$$=\frac{1}{h}\left[\int_0^1 f_i(X+htE_i)hdt\right]$$
$$=\frac{1}{h}\int_0^h g(u)du$$

so u = ht et $g\left(u\right)=f_i\left(X+htE_i\right)$. Using the fundamental integration theorem:

$$\lim_{h\to 0}\frac{1}{h}\int_0^h g(u)du=g(0)$$

because : $$G(h)=\int_0^h g(u)du\Rightarrow G'(h)=g(h)\Rightarrow G'(0)=g(0)$$

but $G'(0)=\lim_{h\to 0}\frac{G(h)-G(0)}{h}=\lim_{h\to 0}\frac{1}{h}\int_0^h g(u)du\Rightarrow G'(0)=g(0)$

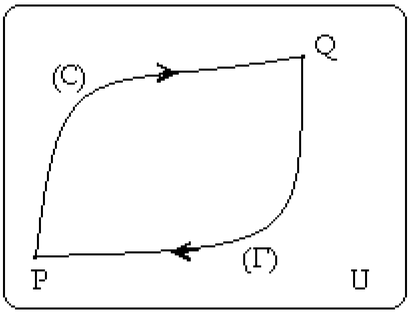
but $g(0)=f_i(X)\Rightarrow D_i\varphi(X)=f_i(X)$ **what needed to be demonstrated**

Theorem 3

Let U be an open connected set, and let F be a vector field on U. If the integral of F around every closed path in U is equal to 0, then F has a potential function in U.

IN FACT

Let P, Q be points in U. Let C and D be paths from P to Q in U. Let D= (D_1, …., D_n) where every Dj is a C^{-1} curve. Then we may form the opposite path D-=(D_n^-, …, D_1^-).



$$\int_C F + \int_{D^-} F = 0 \Rightarrow \int_C F = \int_D F$$

Hence the integral from P to Q is independent of the path. We can then apply theorem 2 to conclude the proof.

Theorem 4

Let F be a vector field defined on the plane from which the origin is deleted, $R^2 - \{(0,0)\}$, and write F=(f, g), verifying:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Let C be a circle of radius 1 and centered at the origin, oriented counterclockwise.

1st case:

If $\int_C F = 0 \Rightarrow$ F has a potential function.

2nd case:

Let $k = \frac{1}{2\pi}\int_C F$ et $G(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

If $k = \frac{1}{2\pi}\int_C F \neq 0 \Rightarrow \exists \Phi : R^2 \to R$ such that:

$$F(x,y) = kG(x,y) + grad\Phi(x,y)$$