Chapter VI

Double integrals

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Part 1

1. Squarable sets

Pave

We shall name **closed boundaries pave** of R² every Cartesian product $P = I \times J$ where *I* and *J* are intervals of R.

$P = [a_1, a_2] \times [b_1, b_2]$ is an example of a closed boundaries pave of R².

Pavable set

We shall name **Pavable set** of R² every union of a finite number of bounded pave of R².

Squarable sets

Let D a bounded set of R². We shall name:

- $A^+(D)$ the lower boundary of the areas of the pavable set of R² containing *D*.
- $A^-(D)$ the upper boundary of the areas of the pavable set of R² contained inside *D*.

We shall say that D is Squarable if and only if $A^-(D) = A^+(D)$; in this case it is named area of D $A(D)$ defined by: $A(D) = A^-(D) = A^+(D)$

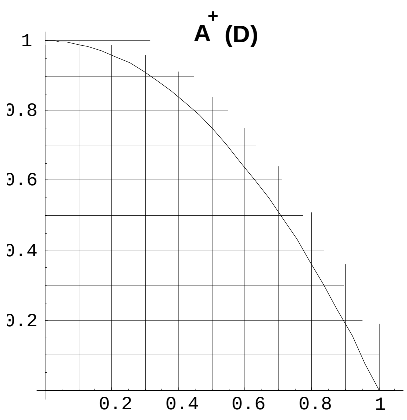


Figure 1

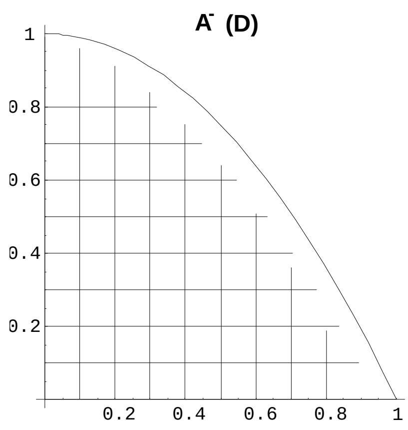


Figure 2

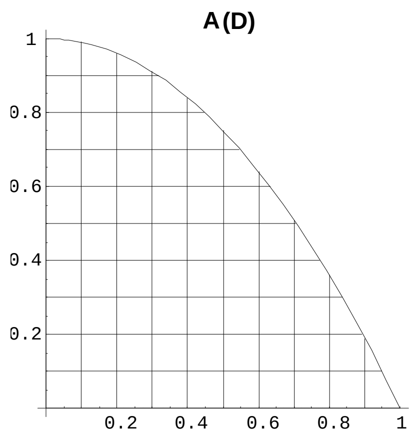


Figure 3

1. Double integral

1.1. Definition

Let f be a continuous function over a rectangle U of R^2. U = [a, b]×[c, d]=I × J. Let's partition I as follow:

$a = x_1 \le x_2 \le \ldots \le x_m = b$ which we also write as $P_I = (x_1, x_2, \ldots, x_m)$.

Similarly let's partition J as follow:

$c = y_1 \le y_2 \le \ldots \le y_n = d$ which we also write as $P_J = (y_1, y_2, \ldots, y_n)$.

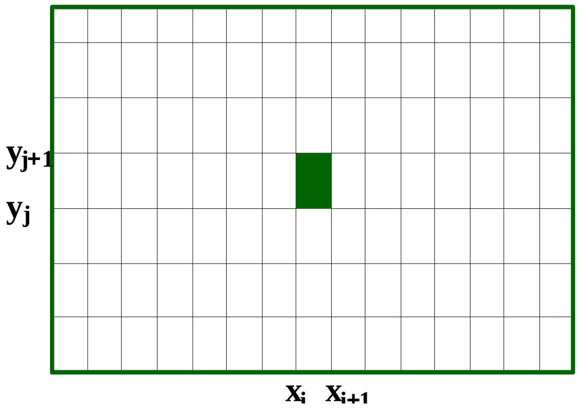


Figure 4

So the rectangle U is partitioned into a multitude of subrectangles:

$$S_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

If U is a rectangle as above, we define its area to be:

Area(U)=(d - c)(b - a) and $\text{Area}(S_{ij}) = (y_{j+1} - y_j)(x_{i+1} - x_i)$

Since f is continuous on U, it is continuous on every subrectangle S_{ij} of U, and since these subrectangles are bounded, the function f admits a minimum $\min_{S_{ij}} f$ and a maximum $\max_{S_{ij}} f$ in each rectangle. Let:

$$W(P, f) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (\max_{S_{ij}} f)(y_{j+1} - y_j)(x_{i+1} - x_i)$$

$$L(P, f) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (\min_{S_{ij}} f)(y_{j+1} - y_j)(x_{i+1} - x_i)$$

If v_{ij} is a point of S_{ij} such that $f\left(v_{ij}\right)$ is nor minimum nor a maximum of the function, so the sum:

$$S(P, f) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(v_{ij})(y_{j+1} - y_j)(x_{i+1} - x_i)$$

is called **Riemann Sum** for f.

Since the lower sums are defined by taking minima and the upper sums are defined by taking the maxima of f over certain rectangles, it's clear that:

$$\min_{S_{ij}} f \le f\left(v_{ij}\right) \le \max_{S_{ij}} f \quad \Rightarrow \quad L(P, f) \le S(P, f) \le W(P, f)$$

We define f to be integrable over U if there exists a unique number which is greater or equal to every lower sum, and less than of equal to every upper sum. If this number exists, we call it the repeated integral of f and denote it by:

$$\iint_U f = \iint_U f(x, y)dxdy$$

Theorem 1

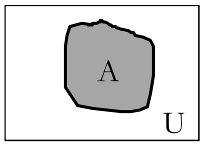
Let U be a rectangle, and let f be a function defined and continuous on U. Then f is integrable on U.

Theorem 2

Let U be a rectangle and let f be a function defined on U, bounded, and continuous except possibly at the points lying on a finite number off smooth curves. Then f is integrable on U.

Proposition

Let A be a region of R^2 which frontier is a curve of class C^1. If A is bounded (contained in a rectangle U) and if f is a continuous function in A the f is integrable on A.



IN FACT

We need to evaluate the integral of f on A. So let's prolong the function f over all the rectangle U by putting:

$$f_U(P) = 0 \qquad \forall P \in U - A$$
$$f_U(P) = f(P) \qquad \forall P \in A$$

The function f_U is continuous on U, except perhaps on the frontier of A which is a curve of class C^1. So let $\iint\limits_A f = \iint\limits_U f_U$

Theorem 3

Every application $f : D \to \mathbb{R}$ bounded and continuous on a squarable set D of R^2 is integrable on D.

1.2. Elementary properties of repeated integrals

Theorem 4

Assume that f, g are functions on the rectangle U, and are integrable. Then f+g is integrable. If k is a number then kf is integrable.

$$\iint_U (f+g) = \iint_U f + \iint_U g; \quad \iint_U kf = k\iint_U f; \quad k \in \mathbb{R}$$

Theorem 5

Assume that f, g are functions on the rectangle U, and are integrable.

If $\forall X \in U;\ f(X) \leq g(X) \Rightarrow \iint_U f \leq \iint_U g$

Theorem 6

Let A be a bounded region in the plane expressed as a union of 2 regions B and C having no point in common except possibly a finite number of curves. If is a function defined on A and continuous except at a finite number of smooth curves then: $\iint_A f = \iint_B f + \iint_C f$

In other words: $A = B \bigcup C$ avec $\overset{\circ}{B} \bigcap \overset{\circ}{C} = \varnothing$

Theorem 7 (Inequality of Cauchy-Schwarz)

Assume that f, g are functions on the rectangle U, and are integrable. Then, f^2 and g^2 are integrable on U and: $\left(\iint_D fg\right)^2 \leq \left(\iint_D f^2\right)\left(\iint_D g^2\right)$

2. Evaluation of repeated integrals

2.1. Evaluation over a pave

Theorem 6

Let U be a rectangle [a, b]×[c, d], and let f be integrable over U. Assume that for each $x \in [a,b]$ $f_x(y) = f(x,y)$ *is integrable over [c, d] then the function of x :* $\int_c^d f(x,y)dy$ *is integrable over [a, b]*

and $\iint_U f = \int_a^b \left(\int_c^d f(x,y)dy\right)dx$

EXAMPLE

Evaluate $\int_1^2 \int_{-3}^4 x^2 ydydx$ sol. $\dfrac{49}{6}$

Particular case

Let

- $(a,b,c,d) \in R^4$ such that $a \leq b$ et $c \leq d$

- $D=[a,b]\times[c,d]$
- $u:[a,b]\to R$ continuous
- $v:[c,d]\to R$ continuous

So the application $f:D\to R$, defined by $f(x,y)=u(x)v(y)$ is integrable over D:

$$\iint_D u(x)v(y)\,dxdy=\left(\int_a^b u(x)\,dx\right)\left(\int_c^d v(y)\,dy\right)$$

We shall say that the repeated integral has separate variables.

2.2. Evaluation over a squarable domain

Fubini theorem

1st Version

Let: $(a,b)\in R^2$ such that $a\le b$

- $g_1,g_2:[a,b]\to R$, 2 functions of class C1 such that $g_1\le g_2$
- $D=\{(x,y)\in R^2; a\le x\le b \text{ et } g_1(x)\le y\le g_2(x)\}$
- $f:D\to R$ continuous

So

- D is squarable
- f is integrable over D
- $\iint_D f=\int_a^b\left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right)dx$

This shows that the repeated integral is nothing but 2 simples, inlayed, integrals.

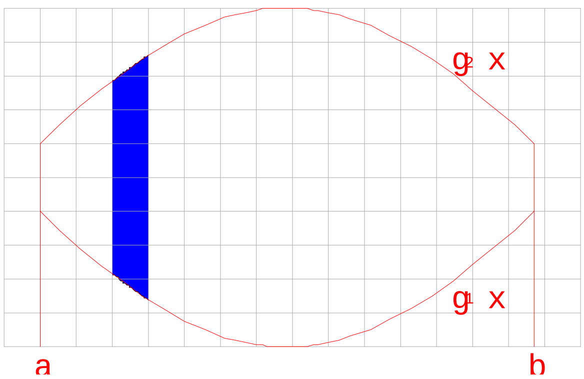


Figure 5

EXAMPLE

1. Evaluate the repeated integral of the function $f(x,y)=x^2+y^2$ over the domain $D=\{(x,y)\ /\ y=x>0,\ y=x^2\}$

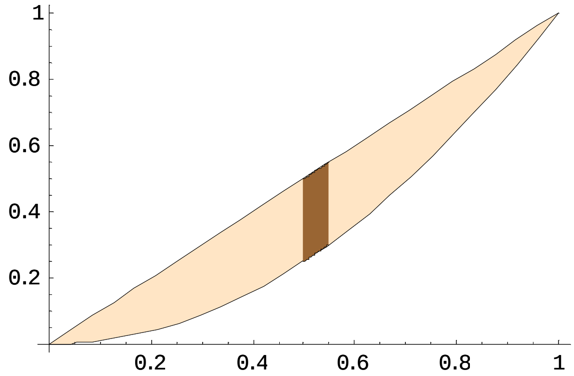


Figure 6

$$\int_0^1\int_{x^2}^{x}\left(x^2+y^2\right)dydx=\int_0^1\left(x^2y+\frac{y^3}{3}\right)_{x^2}^{x}dx=\int_0^1\left[x^2\left(x-x^2\right)+\left(\frac{x^3}{3}-\frac{x^6}{3}\right)\right]=\frac{3}{35}$$

2nd Version

Let:

- $(c,d)\in R^2$ such that $c\le d$
- $g_1,g_2:[c,d]\to R$, 2 functions of class C1 such that $g_1\le g_2$
- $D=\left\{(x,y)\in R^2;c\le y\le d \text{ et } g_1(y)\le x\le g_2(y)\right\}$
- $f:D\to R$ continuous

So

- D is squarable
- f is integrable over D
- $\iint\limits_D f=\int_c^d\left(\int_{g_1(y)}^{g_2(y)}f(x,y)dx\right)dy$

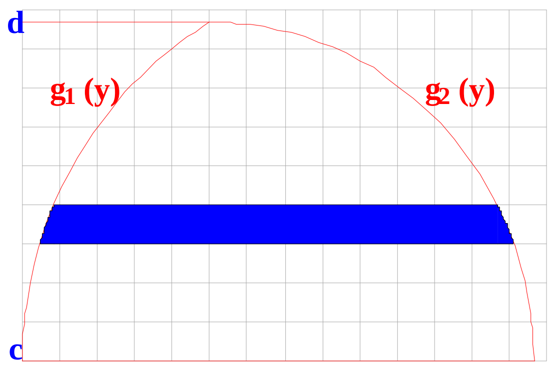


Figure 7

EXAMPLE

2. $f(x,y)=x^2y^2$ and D bounded by y=1, y=2, x=0, x= y. (sol. $7/2$)

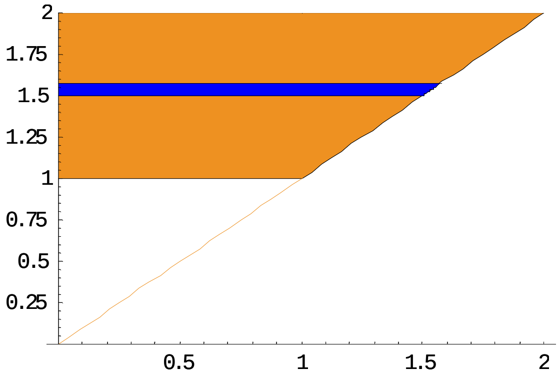


Figure 8

$$\int_1^2\int_0^y x^2y^2dxdy=\int_1^2 y^2\left[\frac{x^3}{3}\right]_0^y dy=\int_1^2\frac{y^5}{3}dy=\frac{7}{2}$$

2.3. Area evaluation

The surface of a domain A of R^2 is given by $S(A)=\iint_A dxdy$

IN FACT

$$S(A)=\iint_A dxdy=\int_a^b\int_{g_1(x)}^{g_2(x)}dydx=\int_a^b(g_2(x)-g_1(x))dx=\int_a^b g_2(x)dx-\int_a^b g_1(x)dx$$

EXAMPLE

Evaluate the area of the domain bounded by the line y = x and the curve $y=x^2$.

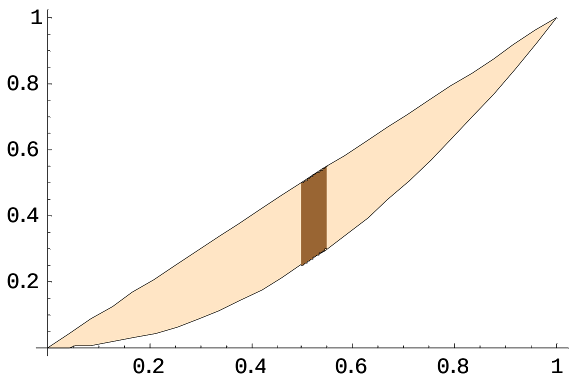


Figure 9

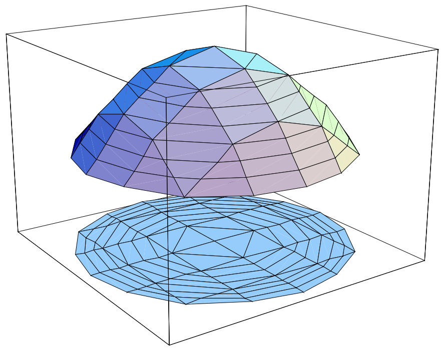
$$\iint_D dydx=\int_0^1\int_{x^2}^x dydx=\frac{1}{6}$$

2.4. Volume evaluation

If the function f(x, y) is positive for every (x, y) of U, and if f(x, y) represents an altitude (z), the repeated integral is the volume of the region of the 3D space above the set U and bounded from the upper part by the graph of f.

In other terms, if f(x, y) is the equation of a surface S, the volume V bounded by the surface S and the plane xOy is given by:

$$V = \iint_U f(x, y)dxdy$$



Example

$$D = \{(x, y) / 0 \le x + y \le 1\}$$

$$\int_0^1 \int_0^{1-x} (1 - x - y)\, dydx \quad \text{Rép.} \frac{1}{6}$$

This integral is the volume bounded by the plane z=1-x-y located above the triangle $0 \le x + y \le 1$ and the plane xOy.

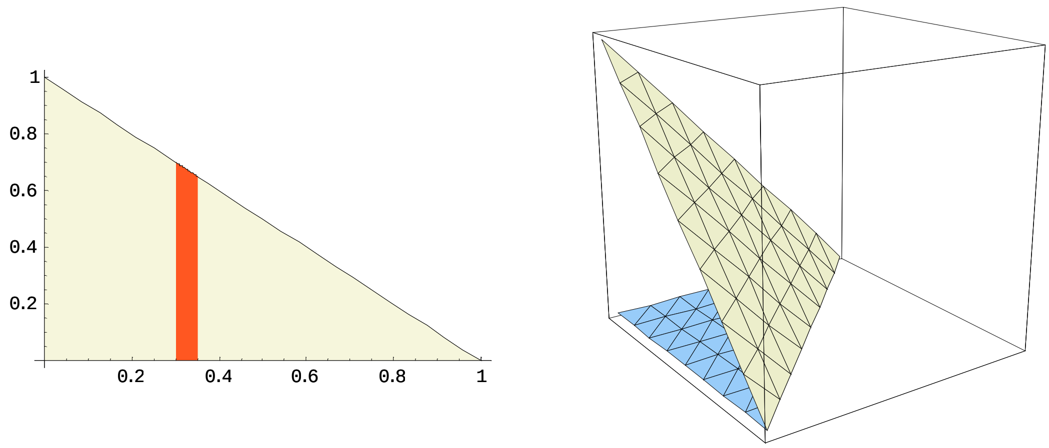


Figure 10

Part 2

3. Variable substitution in a repeated integral

3.1. Ordinary coordinates

Let f be a 2 variables (x and y) function defined on an open set U. Let's assume x = x(u, v) and y = y(u, v) 2 functions of class C^1 on U. The Jacobean of the transformation $(x,y)\to(u,v)$ is by definition given by:

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}x'_u & x'_v\\ y'_u & y'_v\end{vmatrix}=x'_u y'_v - x'_v y'_u$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}$$

We demonstrate that:

$$\iint_S f(x,y)dxdy=\iint_{S^*} f\left(x(u,v),y(u,v)\right)\left|J\right|dudv$$

3.2. Complex function

- We also demonstrate that if J is the Jacobean of the transformation $(x,y)\to(u,v)$ so the Jacobean of the transformation $(u,v)\to(x,y)$ is $\frac{1}{J}$
- Similarly the Jacobean J of the transformation : $(x,y)\to(u,v)\to(s,t)$ is:

$$J=\frac{\partial(x,y)}{\partial(u,v)}\times\frac{\partial(u,v)}{\partial(s,t)}$$

3.3. Simple variable substitution

$$\begin{cases}x=au+bv\\ y=cu+dv\end{cases}\quad \text{with } (a,b,c,d)\in R^4, ad-bc\neq 0$$

$$\Rightarrow\ J=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc\ \Rightarrow\ \iint_D f(x,y)dxdy=\iint_{D^*} f^*(u,v)\left|ad-bc\right|dudv$$

EXAMPLE

Evaluate the repeated integral of $f(x,y)=x$ on the rectangle: $D=[-1,1]\times[-1,1]$

Let's do the following variable substitution

$$\begin{cases} x=-u \\ y=+v \end{cases} \Rightarrow \begin{cases} a=-1; b=0 \\ c=0; d=+1 \end{cases} \Rightarrow J=-1 \Rightarrow |J|=1$$

$$\Rightarrow \begin{cases} D=\{-1\le x\le 1; -1\le y\le 1\} \\ D^*=\{-1\le u\le 1; -1\le v\le 1\} \end{cases}$$

$$\iint_D xdxdy=\iint_{D^*} -u|-1|dudv=\int_{-1}^{1}\int_{-1}^{1}-ududv=\left(\int_{-1}^{1}dv\right)\left(\int_{-1}^{1}-udu\right)=2\times 0=0$$

3.4. Polar coordinates

$$x=r\cos\theta;\quad y=r\sin\theta \Rightarrow \quad J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}=r$$

$$r=\sqrt{x^2+y^2};\ \theta=ArcTan\left(\frac{y}{x}\right)\Rightarrow$$

$$J_1=\frac{\partial(r,\theta)}{\partial(x,y)}=\begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}=\begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix}=\frac{x^2}{r^3}+\frac{y^2}{r^3}=\frac{1}{r}=\frac{1}{J}$$

EXAMPLE

1. Let's evaluate the area of a disc of radius 1: We know that: $S=\iint_D dxdy$

$$\Rightarrow S=\iint_D dxdy=\iint_{D^*}|J|drd\theta=\int_0^{2\pi}\int_0^a rdrd\theta=\int_0^{2\pi}\frac{a^2}{2}d\theta=\pi a^2$$

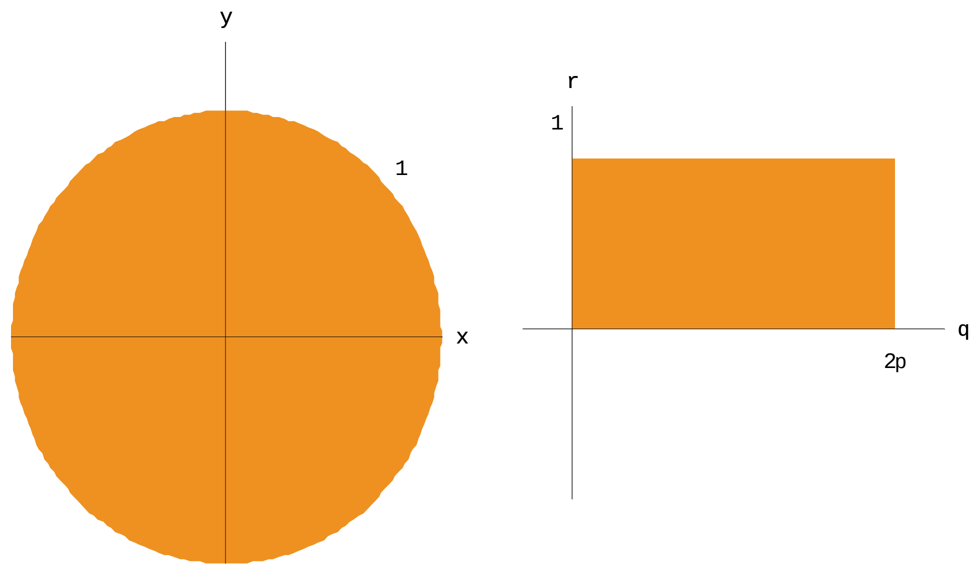


Figure 11

2. Evaluate the repeated integral of the function $f(x,y)=x^2+y^2$ over the domain $D=\{(x,y)\in R^2; x\geq 0, x^2+y^2-2y\leq 0\}$

Using polar coordinates we get:

$$x^2+y^2-2y\leq 0 \Leftrightarrow r^2-2r\sin\theta\leq 0 \Leftrightarrow r\leq 2\sin\theta$$

$$\Rightarrow\ D^*=\left\{(r,\theta)\in R^2; 0\leq\theta\leq\frac{\pi}{2}, 0\leq r\leq 2\sin\theta\right\}$$

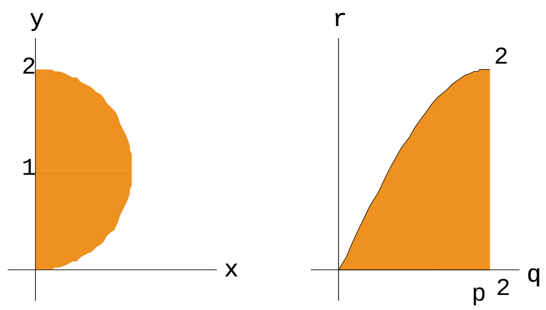


Figure 12

$$\iint_D (x^2+y^2)\,dxdy=\iint_{D^*}(r^2)\,|J|\,drd\theta=\int_0^{\frac{\pi}{2}}\int_0^{2\sin\theta} r^3 drd\theta$$

$$=\int_0^{\frac{\pi}{2}}(1-\cos 2\theta)^2\,d\theta=\frac{3\pi}{4}$$

4. Varied examples

3. f(x, y) = 2xy and D is triangle determined by y = x, x+y=2 et x = 0.

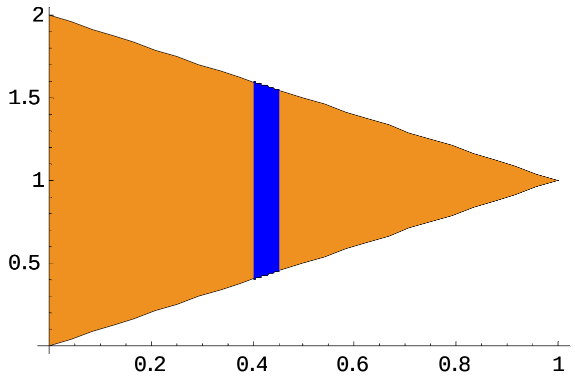


Figure 13

$$\int_0^1 \int_x^{2-x} 2xydydx = \int_0^1 x\left[y^2\right]_x^{2-x} dx = \int_0^1 x(2-2x)\,dx = \frac{2}{3}$$

4. $f(x,y) = |x|, \quad D = \{(x,y)/ -2 \le x \le 1;\ 0 \le y \le |x|\}$ (sol. 3)

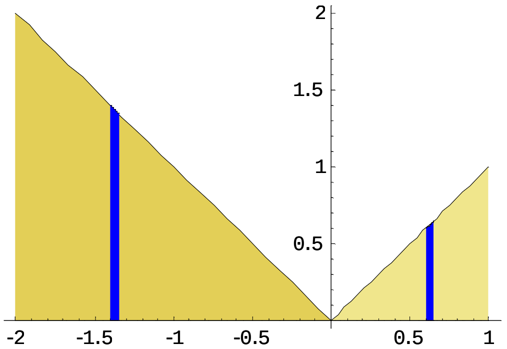


Figure 14

5. $f(x,y) = |y|, \quad D = \{(x,y)/ -2 \le x \le 0;\ |x| \le |y| \le 2\}$ (sol. $16/3$)

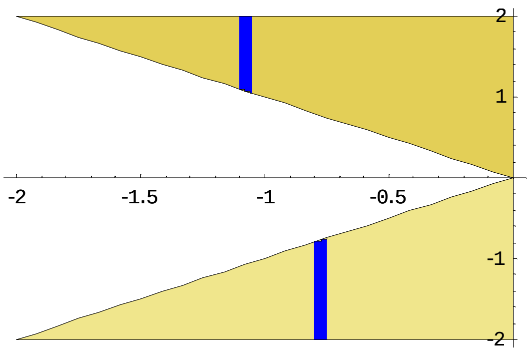


Figure 15

$$\int_{-2}^{0} \int_{-x}^{2} ydydx + \int_{-2}^{0} \int_{-2}^{x} -ydydx = \frac{16}{3}$$

Part 3

5. Repeated integral over a domain having a symmetrical element

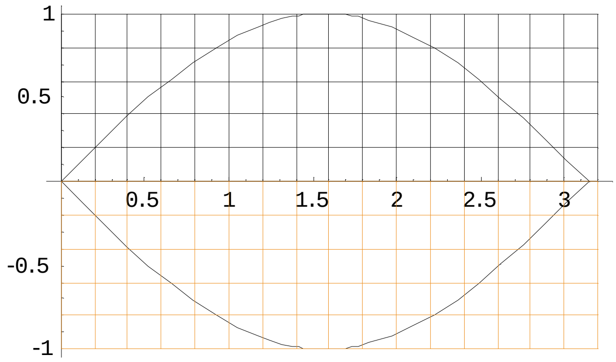
Let's study the following integral:

$$I = \iint_{\Delta} f(x, y)dxdy$$

where Δ is a domain having, a center or an axis of symmetry

a) If the function f associates 2 opposite numbers for 2 symmetrical points P and P' we have I = 0

$$f(P) = -f(P') \Rightarrow I = \iint_{\Delta} f(x, y)dxdy = 0$$



Domain having the x axis as a symmetry line with a function verifying: $f(P) = -f(P')$;

$$I = \int_0^{\pi} \int_{-\sin x}^{\sin x} \frac{xy}{2} dydx = 0$$

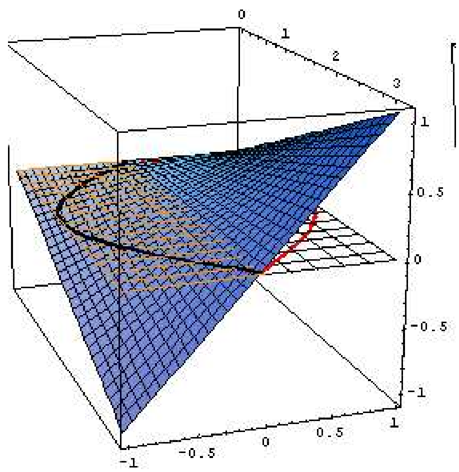


Figure 16

b) If the function f associates 2 equal numbers to 2 symmetrical point P and P' we can similarly write:

$$f(P) = f(P') \Rightarrow I = 2\iint_{\Delta'} f(x, y)dxdy$$

Where Δ' is a part of the domain Δ and Δ'' is its symmetry, the domains Δ' and Δ'' being distinct and Δ'∪Δ''=Δ

Domain having the x axis as an axis of symmetry and a function f verifying

$$f(P) = f(P') \;;$$

$$I = \int_0^{\pi} \int_{-\sin x}^{\sin x} \left| \frac{xy}{2} \right| dydx$$

$$I = 2\int_0^{\pi} \int_0^{\sin x} \left| \frac{xy}{2} \right| dydx = \frac{\pi^2}{8}$$

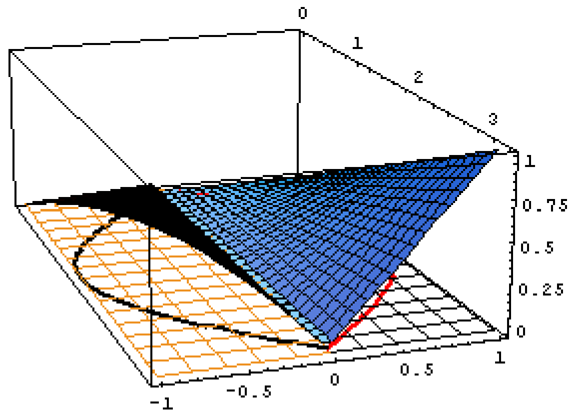


Figure 17

6. Evaluation of the repeated integral using level curves

Theorem

If the integration domain is generated by Δ level curves $f(x, y) = t$ of the function f then:

$$\iint_{\Delta} f(x, y)dxdy = \int_{t_0}^{t_1} tdA(t)$$

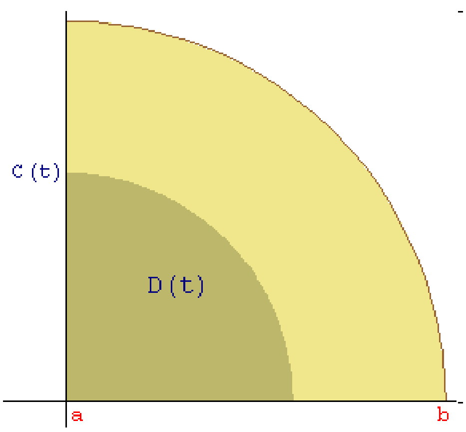
where A(t) is the area of Δ, written as a function of t.

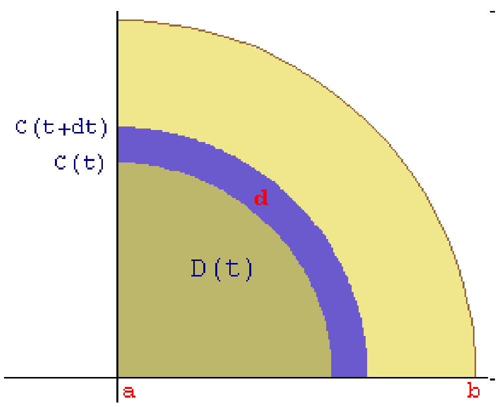
DEMONSTRATION

Let f be a 2 variables function on a domain Δ.

Let C(t) be a curve defined by $f(x, y) = t$. We shall assume that:

- Δ is generated by the curves C(t) when t grows from *a* to *b*.
- D(t) is the part of the domain between C(*a*) and C(t).





$$d = D(t + dt) - D(t)$$

$$\Delta = D(b)$$

Figure 18

Let : $A(t) = \iint_{D(t)} dxdy = aire\,de\,D(t)$ and d = D(t+dt) - D(t)

So : $A(t+dt) - A(t) = \iint_d dxdy =$ area of d

Let : $F(t) = \iint_{D(t)} f(x,y)dxdy$.

So : $F(b) = \iint_\Delta f(x,y)dxdy$ = the integral to evaluate

$$F(a) = 0$$

By applying the Chales relation we can write:

$$F(t+dt) - F(t) = \iint_d f(x,y)dxdy$$

For every (x,y) of « d » we have:

$$t \le f(x,y) \le t + dt$$

$$\Rightarrow \iint_d tdxdy \le \iint_d f(x,y)dxdy \le \iint_d (t+dt)dxdy$$

$$\Rightarrow t\iint_d dxdy \le \iint_d f(x,y)dxdy \le (t+dt)\iint_d dxdy$$

$$\Rightarrow t\left(A(t+dt) - A(t)\right) \le F(t+dt) - F(t) \le (t+dt)\left(A(t+dt) - A(t)\right)$$

$$\Rightarrow t\left(\frac{A(t+dt) - A(t)}{dt}\right) \le \frac{F(t+dt) - F(t)}{dt} \le (t+dt)\left(\frac{A(t+dt) - A(t)}{dt}\right)$$

When dt strives toward zero:

$$F'(t) = tA'(t) \Rightarrow \int_a^b F'(t)dt = \int_a^b tA'(t)dt \Rightarrow F(b) - F(a) = \int_a^b tdA(t)$$

But $F(a) = 0 \Rightarrow F(b) = \int_a^b tdA(t) \Rightarrow F(b) = \iint_\Delta f(x,y)dxdy = \int_a^b tdA(t)$

What needed to be demonstrated

Example 1

Evaluate $\iint_\Delta \left(x^2 + y^2\right)dxdy$ where Δ is the disc $0 < x^2 + y^2 \le a^2$

$$\iint_{\Delta}\left(x^2+y^2\right)dxdy=\int_0^a t^2 d\left(\pi t^2\right)=2\pi\int_0^a t^3 dt=\frac{\pi a^4}{2}$$

Example 2

Evaluate $I=\iint_{\Delta}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy$; Δ defined by $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\le 0$.

The level curves $f(x,y)=t$ generate the domain Δ when t grows from 0 to 1.

A(t) = πabt.

A'(t) = πab

$$I=\int_0^1 t\pi ab dt=\frac{\pi ab}{2}$$

7. Mass, Center of inertia, Moment of inertia

7.1. Mass of a plane plate

We call **plane plate** every couple (D,ρ) where **D** is a squarable part of R^2 and $\rho : D \to R_+$ a continuous application called **surface density** of the plate D.

We call mass of a plane plate (D,ρ) the real number m defined by $m=\iint_D \rho(M)dxdy$, where $M(x,y)$ covers (D,ρ)

Example

Evaluate the mass of a plane pate defined by $f(x)=\sin(x);0<x<\pi$ and having the following surface density $\rho(x,y)=2.5x$.

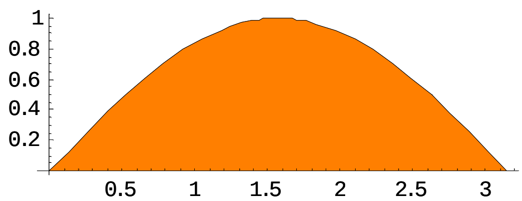


Figure 19

$$m=\iint_D \rho(x,y)dydx=\int_0^{\pi}\int_0^{Sinx}2.5xdydx=\int_0^{\pi}2.5x\sin xdx=\frac{5\pi}{2}$$

7.2. Center of inertia of a plane plate

The center of inertia of a plane plate (D,ρ) of R^2 is the point G of R^2 defined by:

$$x_G = \frac{1}{m}\iint_D x\rho(x,y)\,dxdy$$

$$y_G = \frac{1}{m}\iint_D y\rho(x,y)\,dxdy$$

where (x,y) moves all over D and m the mass of (D,ρ).

Example

Find the center of inertia of the previous plate

$$x_G = \frac{1}{m}\iint_D x\rho(x,y)\,dxdy = \frac{2}{5\pi}\int_0^{\pi}\int_0^{Sinx} 2.5x^2 dydx = \frac{\pi^2-4}{\pi}$$

$$y_G = \frac{1}{m}\iint_D y\rho(x,y)\,dxdy = \frac{2}{5\pi}\int_0^{\pi}\int_0^{Sinx} 2.5xydydx = \frac{\pi}{8}$$

7.3. Moment of inertia of a plane plate

Let H be a point or a line of R^2; for every point M of R^2, we denote $d(M,H)$ the distance from M to H.

The moment of inertia of a line (D,ρ) of R^2 with respect to H is the real number I_H defined by:

$$I_H = \iint_D \rho(M)\left(d(M,H)\right)^2 dxdy$$

where $M(x,y)$ moves all over D .

The moment of inertia with respect to the x axis is given by:

$$I_x = \iint_D y^2\rho(M)\,dxdy$$

The moment of inertia with respect to the y axis is given by:

$$I_y = \iint_D x^2\rho(M)\,dxdy$$

Finally, the moment of inertia with respect to the origin O is given by:

$$I_O = \iint_D (x^2+y^2)\rho(M)\,dxdy$$

Example

Evaluate the moment of inertia with respect to the x axis of the homogenous plate D representing a disc of radius R and centered at the origin.

In polar coordinates the disc becomes: $D = \{0 \le \theta \le 2\pi; 0 \le r \le a\}$

$$I_x = \rho \iint_D y^2 dxdy = \rho \int_0^{2\pi} \int_0^a r^3 \sin^2 \theta dr d\theta = \frac{\pi}{4} a^4 \rho$$