Chapter VII

Green's Theorem

1 The standard version

Green's theorem

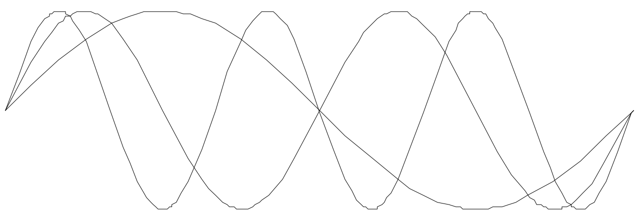
Let P and Q be functions of class C^1 on a region A, which is the interior of a closed path C, parameterized counterclockwise. Then:

$$\int_{C^+} Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

WE SHALL NOTE THAT:

It's difficult to prove green's theorem in general, partly because it's difficult to make rigorous the notion of 'interior' of a path, and also the notion of counterclockwise. In practice, for any specifically given region, it's always easy, however.

Which is the interior of this curve?



1ST CASE

Let's assume that the region is defined by the inequalities:

$$\forall x \quad / \quad a \le x \le b; \Rightarrow g_1(x) \le y \le g_2(x)$$

The boundary of A then consists of four pieces, the two vertical segments x = a (C_3) and x = b (C_4); and the two pieces parameterized by:

$$C_1(t) = (t, g_1(t)); \quad a \le t \le b$$

$$C_2(t) = (t, g_2(t)) \; ; \quad t : b \to a$$

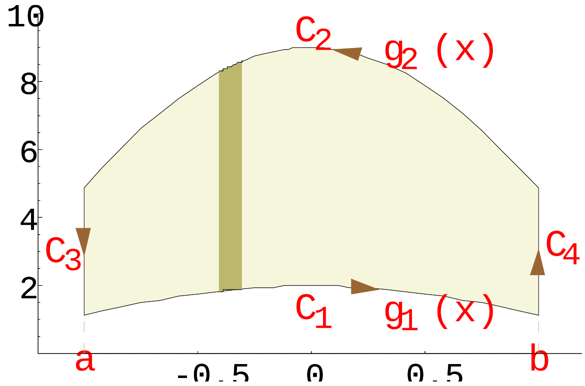


Figure 1

Then we can prove one-half of Green's theorem, namely

$$\int_{C^+} P(x,y)dx = \iint_A -\frac{\partial P(x,y)}{\partial y} dydx$$

PROOF

$$\iint_A -\frac{\partial P(x,y)}{\partial y} dydx = -\int_a^b dx \int_{g_1(x)}^{g_2(x)} \frac{\partial P(x,y)}{\partial y} dy = -\int_a^b [P(x, g_2(x)) - P(x, g_1(x))]dx$$

$$= -\int_a^b P(x, g_2(x))dx + \int_a^b P(x, g_1(x))dx = \int_{C_2} Pdx + \int_{C_1} Pdx$$

However: $x = b \Rightarrow dx = 0 \Rightarrow \int_{C_4} Pdx = 0$

Similarly: $x = a \Rightarrow dx = 0 \Rightarrow \int_{C_3} Pdx = 0$

But the boundary of A consists of: C_1, C_2, C_3, C_4 So:

$$\int_{C^+} Pdx = \int_{C_1} Pdx + \int_{C_2} Pdx + \int_{C_3} Pdx + \int_{C_4} Pdx = \int_{C_2} Pdx + \int_{C_1} Pdx = -\iint_A \frac{\partial P(x,y)}{\partial y} dydx$$

2ND CASE

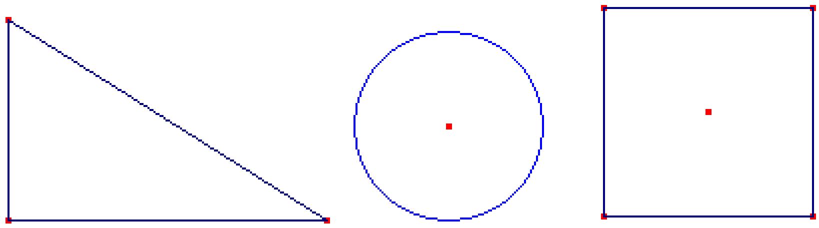
Suppose that the region is given by similar inequalities as in the 1st case, but with respect to the y axis. In other words, the region A is defined by:

$$c \le y \le d; \qquad g_1(y) \le x \le g_2(y)$$

So:

$$\int_{C^+} Q(x,y)dy = \iint_A \frac{\partial Q(x,y)}{\partial x} dxdy$$

In particular, if a region is of a type satisfying both the preceding conditions, then the full theorem follows. Examples of such regions are rectangles, triangles and interior of circles.



EXAMPLE 1

Find the integral of the vector field F(x, y) = (y + 3x, 2y - x) counterclockwise around the ellipse $4x^2 + y^2 = 4$.

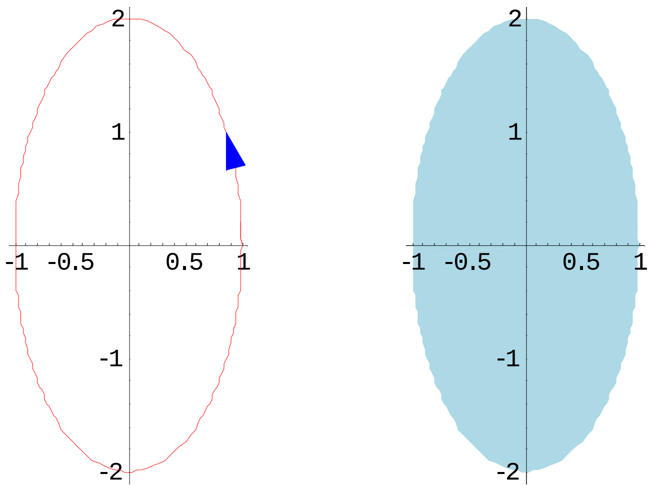


Figure 2

Let: P(x, y)= y + 3x et Q(x, y)=2y - x.

Then $\frac{\partial P}{\partial y} = 1 \ \ et \ \ \frac{\partial Q}{\partial x} = -1$

P and Q being of class C[1] inside the ellipse, we can apply Green's theorem:

$$\int_{C^+} Pdx + Qdy = \iint_A (-2)dydx = -2 \times Aire(A)$$

EXAMPLE 2

Let $F(x,y) = (3xy, x^2)$. Find the integral of F around the rectangle [-1,3]×[0,2], counterclockwise.

P and Q being of class C[1] inside the rectangle, we can apply Green's theorem:

$$\int_{C^+} Pdx + Qdy = \iint_A (2x - 3x)dydx = \int_{-1}^{3} -xdx \int_{0}^{2} dy = -8$$

2 General version

Let A be a region in the plane whose boundary consists of a finite number of curves. Assume that each curve of the boundary is oriented so that A lies to the left of the curve. Let P and Q be functions on A.

Then: $\int_{C^+} Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$

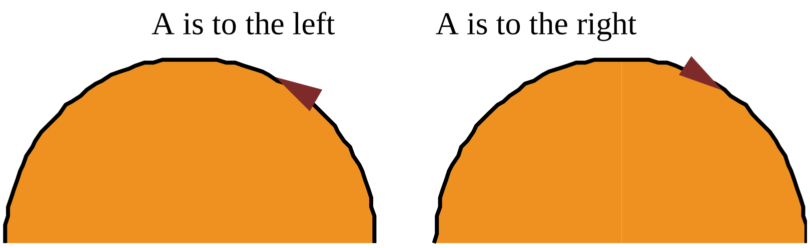


Figure 3

EXAMPLE 3

Let A be the region between two concentric circles as shown, both with counterclockwise orientation.

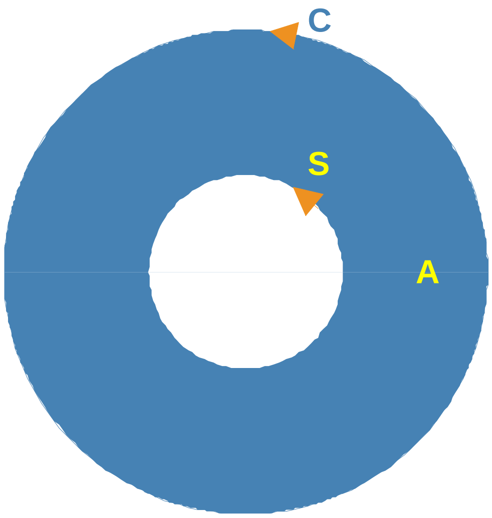


Figure 4

A is at the left of C, but at to the right of S. In order to use Green's theorem we should consider that the boundary Γ consists of {C+, S−}, Where S- is the circle with clockwise orientation.

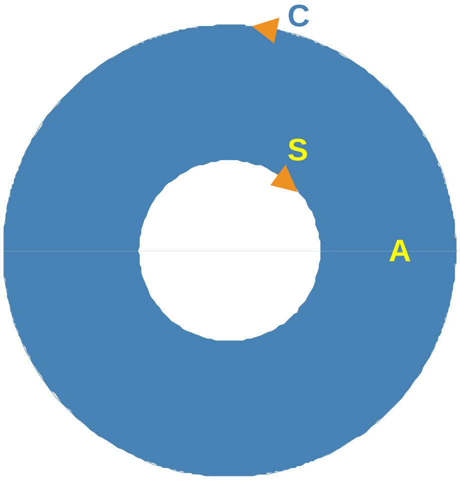


Figure 5

So:

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dydx = \int_{C^+} Pdx + Qdy + \int_{S^-} Pdx + Qdy$$

Remark

If the curve Γ, boundary of a region A, consists of 2 curves C and S, such that one is inside the other and if a field (P,Q) verifies:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

then:

$$\int_{C^+} Pdx + Qdy = \int_{S^+} Pdx + Qdy$$

EXAMPLE 4

Let $G(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$, we need to evaluate the integral of G around S in the counterclockwise direction.

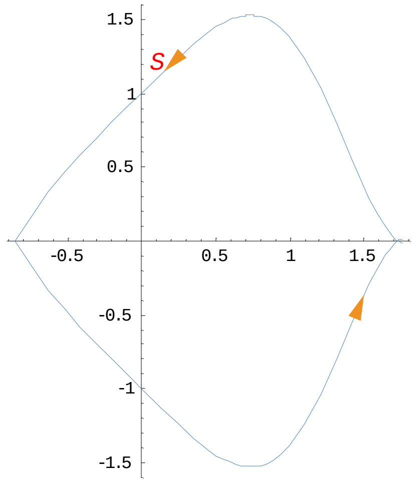


Figure 6

This is not simple. But G verifies:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

We draw a small circle around the origin 0 oriented such that A is at its left.

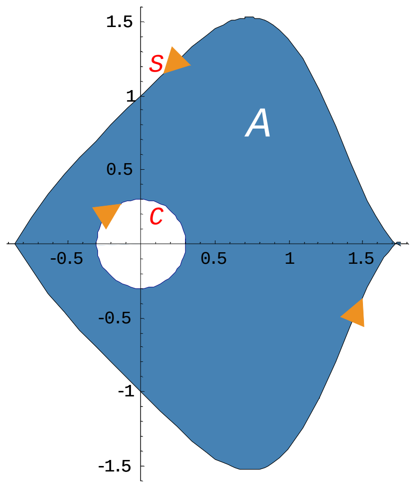


Figure 7

Using Green's theorem we can get:

$$0=\int_{S^+}G+\int_{C^-}G\Rightarrow\int_{S^+}G=\int_{C^+}G$$

But $\int_{C^+}G=2\pi$, already evaluated. $\Rightarrow\int_{S^+}G=2\pi$

3 The divergence and the rotation of a vector field

3.1 Rotation

Let $F(x,y)=(P(x,y),Q(x,y))$ and C a curve of the plane (xOy). Let u be the unit vector held by the tangent.

$$\iint_A\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_{C^+}\mathrm{F.uds}$$

$$\iint_A rotF(x,y)dxdy=\int_{C^+}\mathrm{F.uds}$$

PROOF:

We already know that $ds=\left\|C'(t)\right\|dt$

$$\Rightarrow u(t)=\frac{C'(t)}{\left\|C'(t)\right\|}\Rightarrow C'(t)=u(t)\left\|C'(t)\right\|$$

$$\Rightarrow C'(t)dt=u(t)\left\|C'(t)\right\|dt=u(t)ds$$

$$\Rightarrow\int_{C^+}F(C(t)).C'(t)dt=\int_{C^+}F(C(t)).u(t)ds$$

By applying Green's theorem:

$$\iint_A\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\int_{C^+}F(C(t)).C'(t)dt=\int_{C^+}F.uds$$

3.2 Divergence

Let $F(x,y,z)=(P,Q,R)(x,y,z)$. We shall call divergence of F the quantity denoted by divF, where:

$$divF=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

If the field is defined on R² by $F(x,y)=(P(x,y),Q(x,y))$ we will get:

$$divF=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$$

Let n be a unit vector which is normal to the curve C. Then:

$$\iint_A divFdxdy=\int_{C^+} F.nds$$

PROOF:

$$C'(t)=(x'(t),y'(t))\Rightarrow N(t)=(y'(t),-x'(t))\Rightarrow \|N(t)\|=\|C'(t)\|$$

$$\Rightarrow n(t)=\frac{N(t)}{\|N(t)\|}\Rightarrow N(t)=n(t)\|N(t)\|$$

$$\Rightarrow N(t)dt=n(t)\|N(t)\|dt=n(t)\|C'(t)\|dt=n(t)ds$$

So:

$$\iint_A\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)=\int_{C^+}(-Q,P).(x',y')dt$$

$$=\int_{C^+}(P,Q).(y',-x')dt$$

$$=\int_{C^+}F.N(t)dt$$

$$=\int_{C^+}F.n(t)ds$$