Chapter VIII

Triple integrals

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Wissam Karam

PART 1

1 Triple integrals

1.1 Definition

Let f be a continuous function in a 3D rectangular box B of R^3.

$$B=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$$

The volume of B is given by:

$$V=(b_1-a)_1\times(b_2-a_2)\times(b_3-a_3)$$

A partition P of B is the determined by partitions P_1, P_2, P_3

$[a_1,b_1]$, $[a_2,b_2]$ and $[a_3,b_3]$.

This partitions B into 3D subrectangles, which we denote by S. Like it has been done for the double integral, we shall define:

$$I(P,f)=\sum_S(\min_S f)Vol(S)$$

$$K(P,f)=\sum_S(\max_S f)Vol(S)$$

f being a bounded function on B. f is called integrable if there exists a unique number which greater than $I(P,f)$ and smaller than $K(P,f)$. If this is the case, this number is called the integral of f and is denoted by:

$$\iiint_B f=\iiint_B f(x,y,z)\,dxdydz$$

1.2 Proprieties

The same theorems of “double integral” chapter are still valid here. We repeat them:

$$\iiint_B (f+g)=\iiint_B f+\iiint_B g\;; \qquad \iiint_B kf=k\iiint_B f$$

Let B be a 3D rectangular box, and let f be a function defined on B, bounded and continuous except possibly at the points lying on a finite number of smooth surfaces. Then f is integrable on B.

If A denotes a 3D region and f is a function on A, we define:

$$\begin{cases} f^*(X)=f(X) & \text{if } x\in A \\ f^*(X)=0 & \text{if } x\in B-A \end{cases}$$

Then

$$\iiint_A f = \iiint_B f^*$$

$$\iiint_V f = \iiint_{V_1} f + \iiint_{V_2} f; \quad V = V_1 \cup V_2; \quad \mathrm{Vol}(V_1 \cap V_2) = 0$$

Remark

We shall divide the 3D space into 8 octants. The annotation is done in the trigonometrical direction.

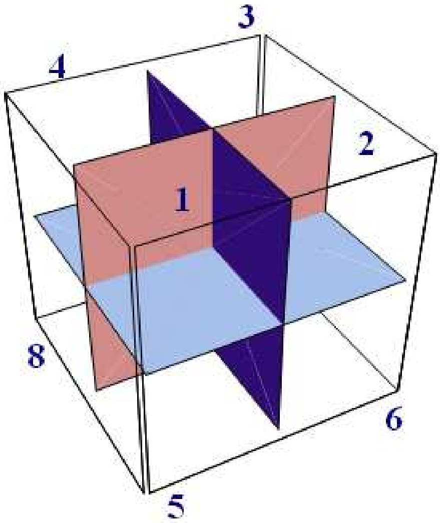


Figure 1

1.2.1 Computation techniques

CASE OF A RECTANGULAR BOX

Let $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

$$\Rightarrow \iiint_B f = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x,y,z)\,dz \right) dy \right] dx$$

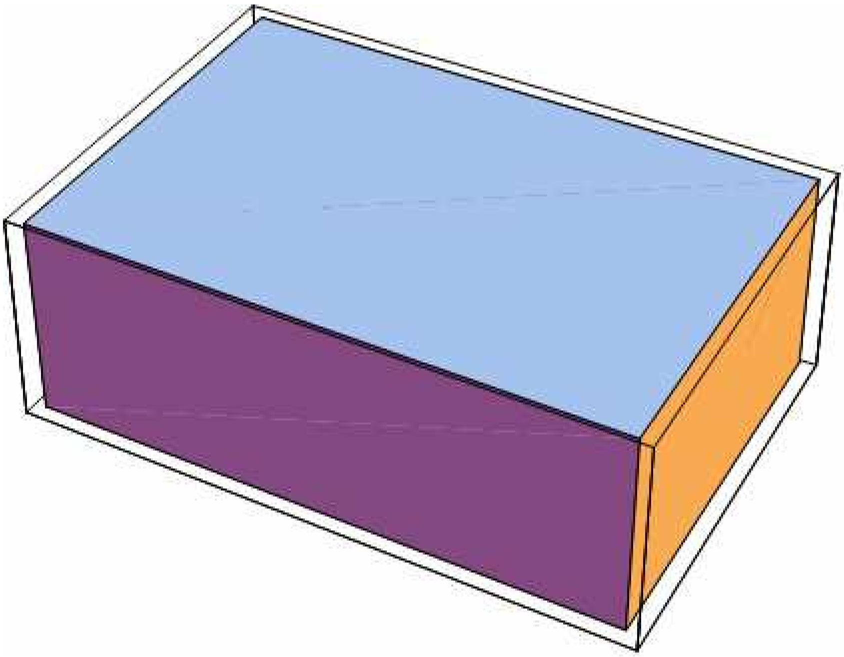


Figure 2

GENERAL CASE

Let a function defined over a domain B as follows:

- a and b are 2 real numbers such that a < b
- $g_1(x)$ and $g_2(x)$ 2 functions defined over [a, b] such that $g_1(x) \le g_2(x)$
- $h_1(x,y)$ and $h_2(x,y)$ 2 functions defined over $[a,b]\times[g_1(x),g_2(x)]$ such that: $h_1(x,y) \le h_2(x,y)$

Then $\iiint_B f = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz \right) dy \right] dx$

We can simply prove that:

$$\iiint_B f = \iint_D dxdy \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz$$

where D is the projection of B on the plane xOy.

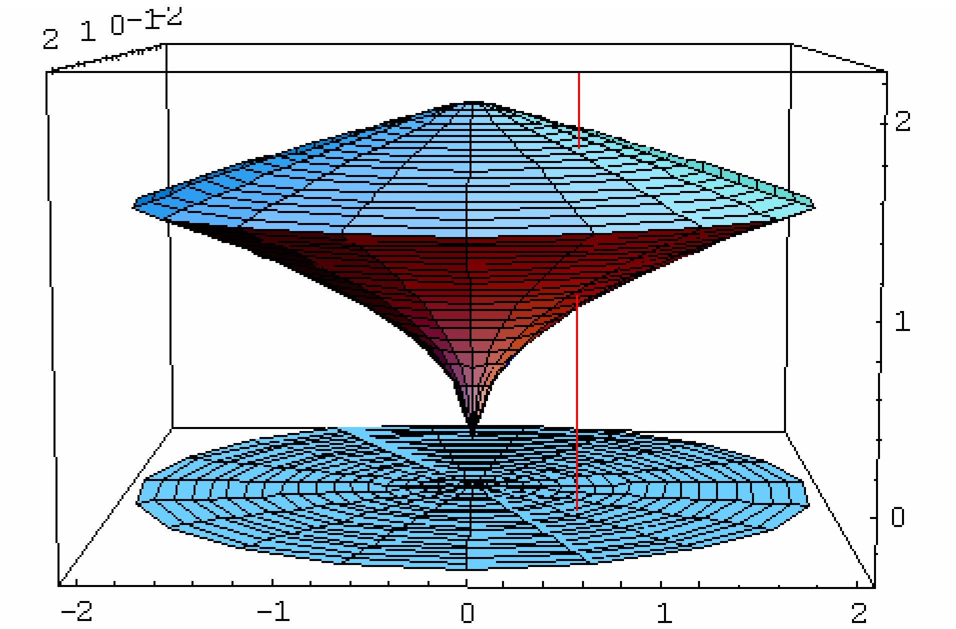


Figure 3

1.3 Volume evaluation

1.3.1 Any domain

when $f = 1$ the triple integral of f over a domain D is the volume of D.

$$V(D) = \iiint_D dxdydz$$

EXAMPLE 1

Evaluate the volume of the tetrahedral defined by $x > 0, y > 0, z > 0$ et $x + y + z < a$.

Method #1

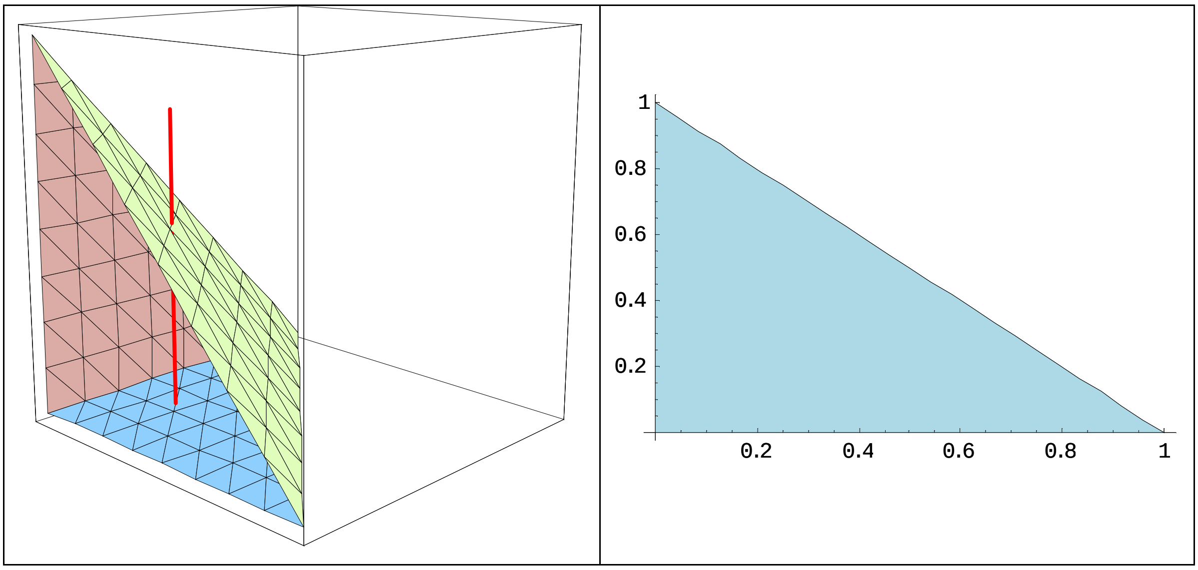
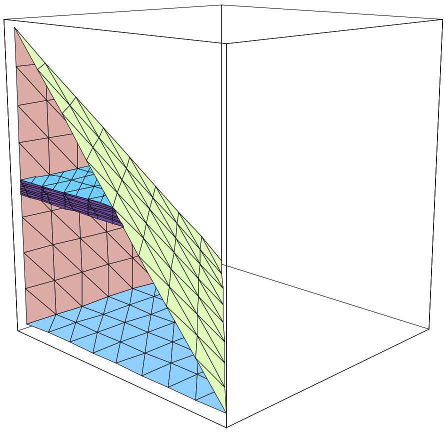


Figure 4

$$V=\iiint_A dxdydz=\iint_D dxdy\int_0^{a-x-y}dz=\int_0^a dx\int_0^{a-x}(a-x-y)\,dy=\frac{a^3}{6}$$

Method #2



$$V=\iiint_A dxdydz=\int_0^a dz\int_0^{a-z}dy\int_0^{a-y-z}dx=\int_0^a\frac{1}{2}(a-z)^2\,dz=\frac{a^3}{6}$$

We note that $V=\int_0^a A(z)\,dz$, where $A(z)$ is the area of the triangle at the altitude z.

1.3.2 Volume of domains with known base surface

The volume of a solid that's section between the planes z = a and z = b have an area $A(z)$, is:

$$V=\int_a^b A(z)dz$$

EXAMPLE

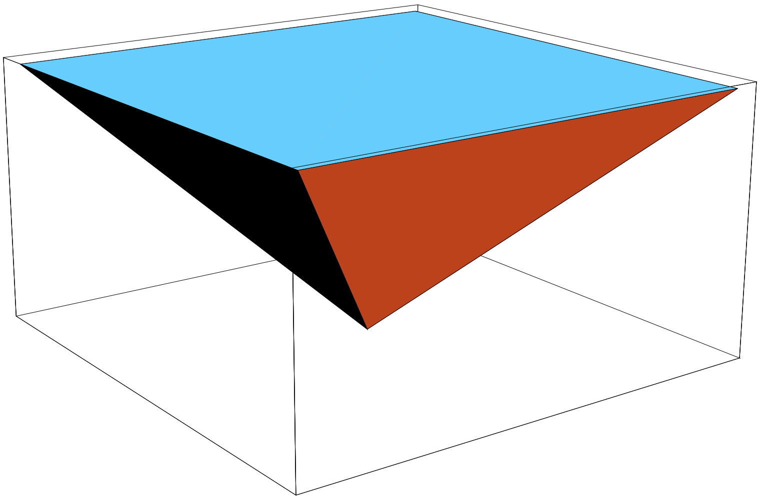


Figure 5

Evaluate the volume of the pyramid having a square base formed by the following points {1,1,1} ; {1,-1,1} ; {-1,-1,1} ;{1,-1,1}

The base is at an altitude z=1 and it's area equals 2×2=4. If we cut this pyramid with a plane parallel to xOy, having an altitude z, the section is also square and its area is $A(z) = 4z^2$

The pyramid volume is therefore: $V = \int_0^1 A(z)\,dz = \int_0^1 4z^2 dz = \frac{4}{3}$

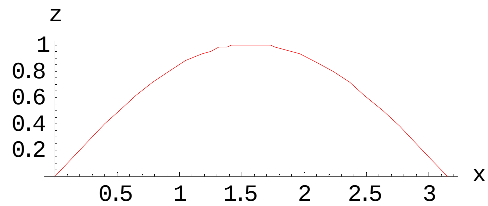
1.3.3 Volume of revolution

The volume obtained by turning the curve {z=f(x) a < x < b}, around the x axis is:

$$V = \int_a^b \pi f(x)^2 dx$$

EXAMPLE

The volume obtained by turning the curve $z = \sin x; \quad 0 \le x \le \pi$ around the x axis.



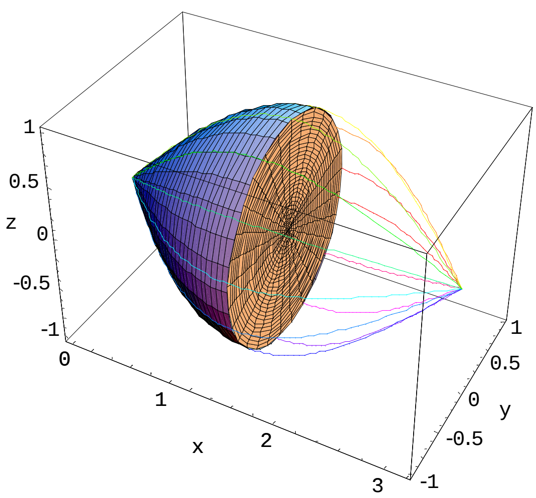


Figure 6

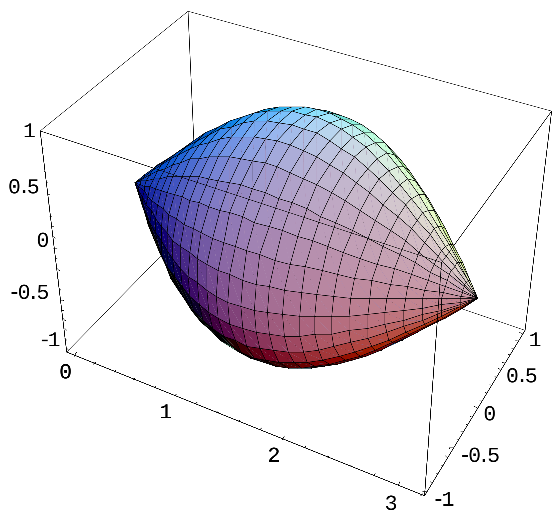


Figure 7

$$V = \int_a^b \pi f(x)^2 dx = \int_0^\pi \pi \sin^2 x dx = \frac{\pi^2}{2}$$

2 Cylindrical coordinates

A point M of the 3D space can be defined by cylindrical coordinates as follows:

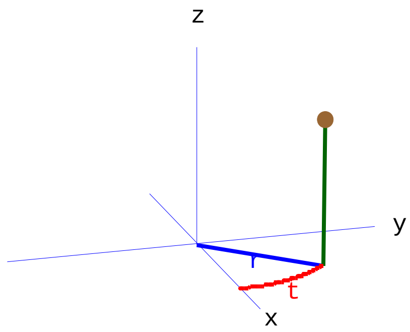


Figure 8

$$x = r\cos\theta;\quad y = r\sin\theta;\quad z = z$$

The Jacobean of this variables substitution is:

$$J = \begin{vmatrix} x'_r & x'_\theta & x'_z \\ y'_r & y'_\theta & y'_z \\ z'_r & z'_\theta & z'_z \end{vmatrix} = r$$

3 Spherical coordinates

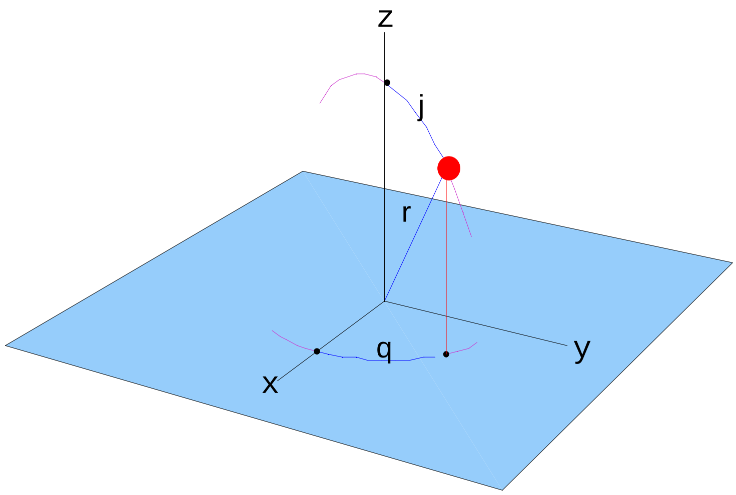


Figure 9

A point M of the 3D space can be defined by spherical coordinates as follows:

$$x = \rho\sin\varphi\cos\theta;\quad y = \rho\sin\varphi\sin\theta;\quad z=\rho\cos\varphi$$

This coordinates are given for:

$$0 \le \rho;\qquad 0 \le \varphi \le \pi;\qquad 0 \le \theta \le 2\pi$$

The Jacobean of this variables substitution is:

$$J = \begin{vmatrix} x'_\rho & x'_\theta & x'_\varphi \\ y'_\rho & y'_\theta & y'_\varphi \\ z'_\rho & z'_\theta & z'_\varphi \end{vmatrix} = \rho^2 \sin\varphi$$

EXAMPLE 3

Evaluate the volume between the cone $z^2 = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = z$.

SOLUTION

This sphere is centered at the point $\left(0, 0, \frac{1}{2}\right)$ and have a radius of $\frac{1}{2}$;

$$x^2 + y^2 + z^2 = z \Rightarrow x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

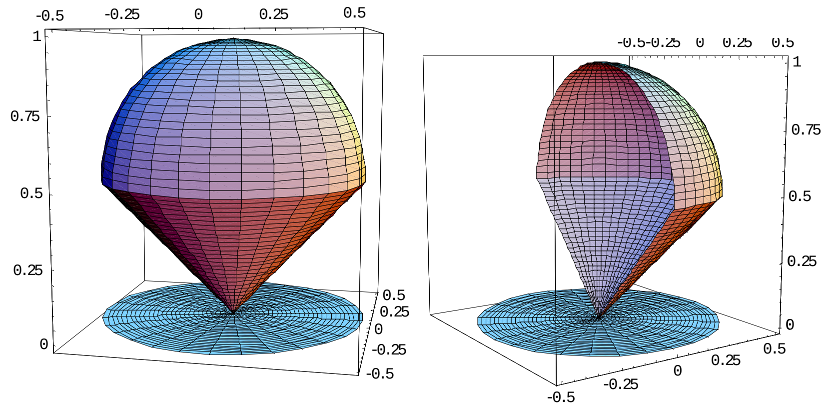


Figure 10

In spherical coordinated the sphere becomes:

$$x^2 + y^2 + z^2 = z \Rightarrow \rho^2 = \rho\cos\varphi \Rightarrow \rho = \cos\varphi$$

So

$$V = \iiint_D dxdydz = \iiint_{D^*} \rho^2 \sin\varphi d\rho d\varphi d\theta$$

$$V = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi d\varphi \int_0^{\cos\varphi} \rho^2 d\rho = \frac{\pi}{8}$$

PART 2

4 Mass, Centre of inertia, Moment of inertia

4.1 Mass of a solid

We shall call a solid every pair (S,ρ) where **S** is a part of R^3 and $\rho: S \to R_+$ a continuous application called space density of the solid (S,ρ).

We shall call mass of a **solid** (S,ρ) the real number **m** defined by
$m = \iiint_S \rho(x,y,z)\,dxdydz$, where (x,y,z) covers S

4.1.1 Example

Evaluate the mass of the solid having the following density $\rho = 3r$ and defined by:

$$-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}; \quad r = \cos\theta; \quad 0 \le z \le r$$

STUDY OF THE SOLIDE:

- $0 \le z \le r \Leftrightarrow 0 \le z \le \sqrt{x^2+y^2}$ $\Rightarrow$ the cone is its upper frontier
- $r = \cos\theta \Rightarrow r^2 = r\cos\theta \Rightarrow x^2+y^2-x=0 \Rightarrow \left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$ $\Rightarrow$ the circular part $\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$ with $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ is its lateral frontier

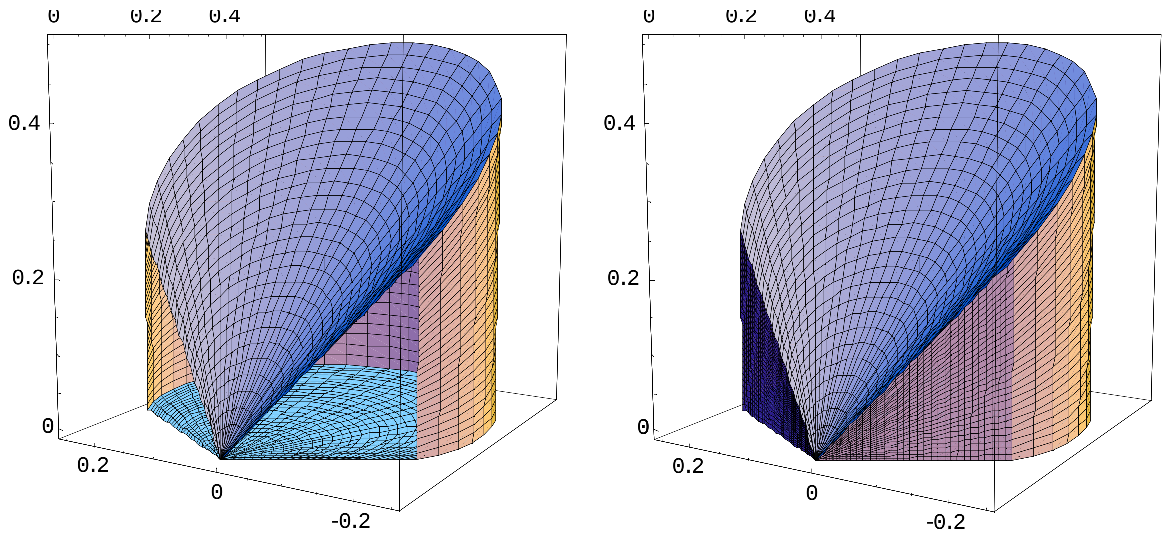


Figure 11

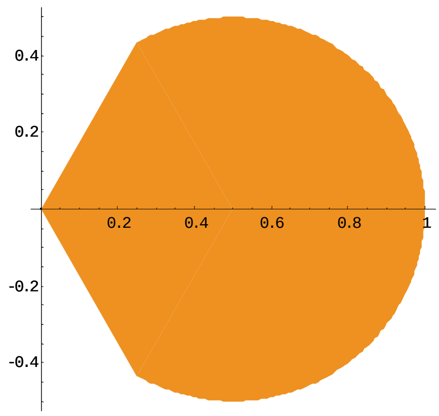


Figure 12

SOLUTION

The masse of M is given by:

$$M = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \int_0^{\cos\theta} 3rdr \int_0^{r} rdz = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \int_0^{\cos\theta} 3r^3 dr = \frac{3}{4}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^4\theta\, d\theta$$

But $\cos^4\theta = \frac{3}{8} + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}$

$\Rightarrow$

$$M = \frac{3}{16}\left[3\theta + 2\sin 2\theta + \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{3}} = \frac{3}{16}\left[\pi + 2\sin\frac{2\pi}{3} + \frac{1}{4}\sin\frac{4\pi}{3}\right] = \frac{3}{16}\left[\pi + \frac{7\sqrt{3}}{8}\right]$$

4.1.2 Example

Evaluate the mass of a ball S, centered at the origin and having a radius R, the density being given by

$$\rho(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

Using spherical coordinates:

$$m = \iiint_S \rho(x,y,z)\,dxdydz = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{R} \rho^3 d\rho = 2\pi \times 2 \times \frac{R^4}{4} = \pi R^4$$

4.2 Center of inertia of a solid

The center of inertia of a solid (S,ρ) is the point G of R^3 defined by:

$$\begin{cases} x_G = \dfrac{1}{m}\iiint\limits_S x\rho(x,y,z)\,dxdydz \\ y_G = \dfrac{1}{m}\iiint\limits_S y\rho(x,y,z)\,dxdydz \\ z_G = \dfrac{1}{m}\iiint\limits_S z\rho(x,y,z)\,dxdydz \end{cases}$$

where (x,y,z) covers S and m is the mass of (S,ρ).

4.2.1 Example

Find the center of gravity of the hemisphere:

$$z = \sqrt{a^2 - x^2 - y^2}$$

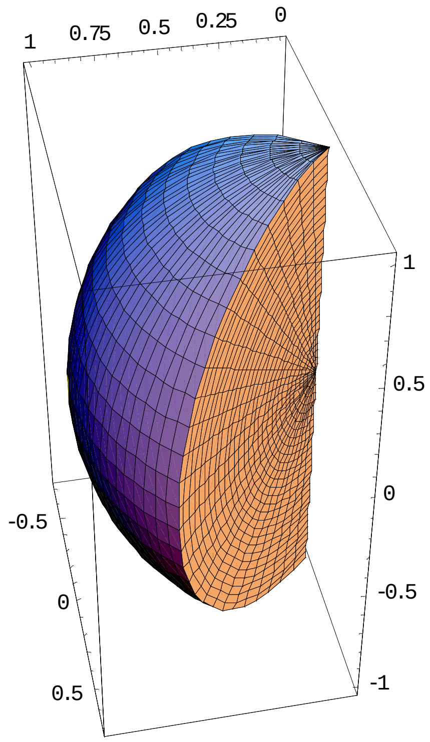
having a constant density of 1. we can simply verify that, due to the symmetry, $\bar{x} = \bar{y} = 0$.

$$z_G = \frac{1}{m}\iiint\limits_D zdxdydz = \frac{3}{2\pi}\iint\limits_D dxdy \int\limits_0^{\sqrt{a^2-x^2-y^2}} zdz$$
$$= \frac{3}{4\pi}\iint\limits_D \left(a^2 - x^2 - y^2\right)^2 dxdy =$$
$$= \frac{3}{4\pi}\int_0^{2\pi}\int_0^a \left(a^2 - r^2\right)^2 drd\theta = \frac{4}{5}a^5$$

4.2.2 Example

Find the center of gravity G of the homogeneous solid (S,ρ) where S is a part of the sphere, centered at the origin and having a radius of 1. It is defined in spherical coordinates by:

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}; 0 \le \varphi \le \pi; 0 \le \rho \le 1$$

 Figure 13	It has the form of a slice of an orange.

SOLUTION

Using the proportionality, the masse m is given by:

$$m = \iiint_S \rho dxdydz = \rho \frac{4/3\,\pi}{2\pi} \times \frac{2\pi}{4} = \frac{\pi\rho}{3}$$

Due to symmetrical reasons $y_G = z_G = 0$

$$x_G = \frac{3}{\pi\rho} \iiint_S \rho x dxdydz$$

Then: $$= \frac{3}{\pi}\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta d\theta\right)\left(\int_0^{\pi} sin^2\varphi d\varphi\right)\left(\int_0^1 \rho^3 d\rho\right)$$

$$= \frac{3\sqrt{2}}{8}$$

4.3 Moment of inertia of a solid

Let H be a point of a line or a plane of R^3; for every point M of R^3, we denote $d(M,H)$ as the distance from M to H.

The moment of inertia of a solid (S,ρ) with respect to H is the real number I_H defined by:

$$I_H = \iiint_S \rho(M)\left(d(M,H)\right)^2 dxdydz$$

where $M\left(x,y,z\right)$ covers S.

IN PARTICUAR:

The moments of inertia of a solid with respect to the 3 main axis (x, y , z) are given by:

$$\begin{cases} I_x = \iiint\limits_A (y^2 + z^2)\rho(x,y,z)dxdydz \\ I_y = \iiint\limits_A (z^2 + x^2)\rho(x,y,z)dxdydz \\ I_z = \iiint\limits_A (x^2 + y^2)\rho(x,y,z)dxdydz \end{cases}$$