# Chapter IX

## Surface integrals

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# Part 1

## 1. Parameterization, tangent plane and normal vector

### 1.1 Parameterized surface

Let's first recall that a parameterized curve is an application from R into $R^2$ or $R^3$;

C(t) =(x(t), y(t)) or C(t)=(x(t), y(t), z(t)), $t \in [a,b]$

$$C(t) = \left(t, \cos\left(t^2\right), 3t+2\right); \quad 0 \le t \le 2\pi$$

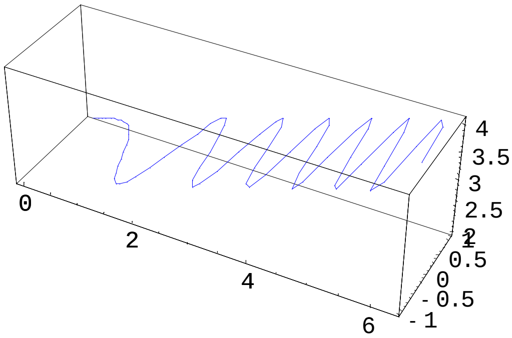



Figure 1

A parameterized surface is an application from $R^2$ into $R^3$

X(t, u)=(x(t, u), y(t, u), z(t, u)) (t, u)$\in$ A $\subseteq . R^2$

$$X(t,v) = \left(t+v, \cos\left(t^2\right)+2v, v^3+1\right); \quad 0 \le t \le \pi; \quad 0 \le v \le 1$$

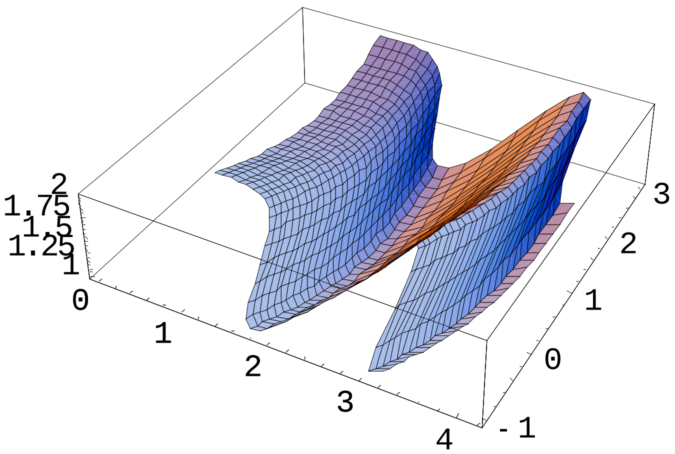



Figure 2

### EXAMPLE

The equation of the sphere centered at the origin having a radius a (a > 0) is in spherical coordinates:

$$X(\theta,\varphi) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi);\ 0 \le \varphi \le \pi; 0 \le \theta \le 2\pi$$

### TANGENT PLANE

Let's consider a surface S, defined by:

X(t, u)=(x(t, u), y(t, u), z(t, u)) (t, u)$\in$ A $\subseteq . R^2$.

For every fixed value of u, the curve:

$$C_1(t) = X(t,u)$$

is a curve from the surface S.

Similarly, for every fixed t, the curve:

$$C_2(u) = X(t,u)$$

is also a curve from the surface S. The vector

$$C_1^{'}(t) = \frac{\partial X}{\partial t}(t,u)$$

is the direction vector of the tangent to the curve $C_1(t)$, therefore it's the direction vector of the tangent to the surface S at point X(t, u).

Similarly, the vector

$$C_2^{'}(u) = \frac{\partial X}{\partial u}(t,u)$$

is the direction vector of the tangent to the curve $C_2(t)$, therefore it's the direction vector of the tangent to the surface S at point X(t, u). So we can propose the following:

**Definition**

*For every surface S defined as above, the plane passing through point X(t, u) and parallel to the 2 vectors* $\frac{\partial X}{\partial t}(t,u)$ *and* $\frac{\partial X}{\partial u}(t,u)$ *(whenever they exist) is by definition the plane tangent to the surface S at point (t, u)*

## 1.2 Normal vector

The vector that is normal to the surface S at point X(t, u) is by definition the normal vector to the tangent plane at this point.

$$N = \frac{\partial X}{\partial t}(t,u) \wedge \frac{\partial X}{\partial u}(t,u)$$

The unit normal vector is then:

$$n = \frac{\frac{\partial X}{\partial t}(t,u) \wedge \frac{\partial X}{\partial u}(t,u)}{\left\| \frac{\partial X}{\partial t}(t,u) \wedge \frac{\partial X}{\partial u}(t,u) \right\|}$$

**EXAMPLE**

Let's consider the following sphere:

$$X(\theta,\varphi) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi); 0 \le \varphi \le \pi; 0 \le \theta \le 2\pi$$

Then we have:

$$\left.\begin{array}{l} \frac{\partial X}{\partial\varphi}(\varphi,\theta) = (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi) \\ \frac{\partial X}{\partial\theta}(\varphi,\theta) = (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0) \end{array}\right\} \Rightarrow \frac{\partial X}{\partial\varphi} \wedge \frac{\partial X}{\partial\theta} = a\sin\varphi\, X(\varphi,\theta)$$

$$\left\| \frac{\partial X}{\partial\varphi} \wedge \frac{\partial X}{\partial\theta} \right\| = a\sin\varphi \, \|X(\varphi,\theta)\| = a^2\sin\varphi \qquad \Rightarrow n = \frac{1}{a} X(\varphi,\theta)$$

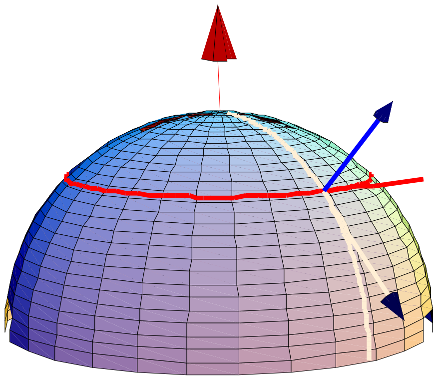


Figure 3

**REMARK 1**

Let's find the gradient to this sphere using the gradient. The sphere's function is given by:

$$f(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$$

$$gradf = 2(x,y,z) \Rightarrow n = \frac{2(x,y,z)}{2\sqrt{x^2+y^2+z^2}} = \frac{2(x,y,z)}{2a} = \frac{1}{a}(x,y,z)$$

But $X(\varphi,\theta)$ and (x, y, z) are nothing but the radius vector OM.

**REMARK 2**

$\frac{\partial X}{\partial \theta} \wedge \frac{\partial X}{\partial \varphi}$ is also a vector normal to the surface, but with a direction opposite to the first one. We shall choose for n the vector directed toward the exterior. If S is not a closed surface then n is directed with respect to the "bonhomme d'Ampère".

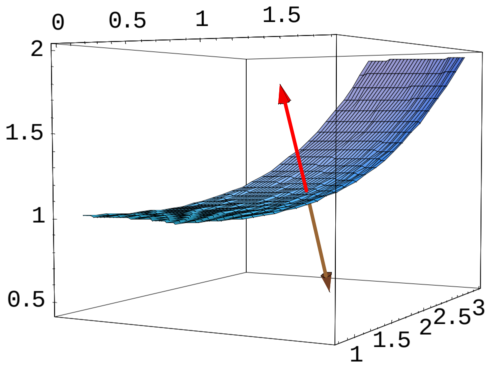



Figure 4

# 2. Area of a parameterized surface

## 2.1 Parameterized surface

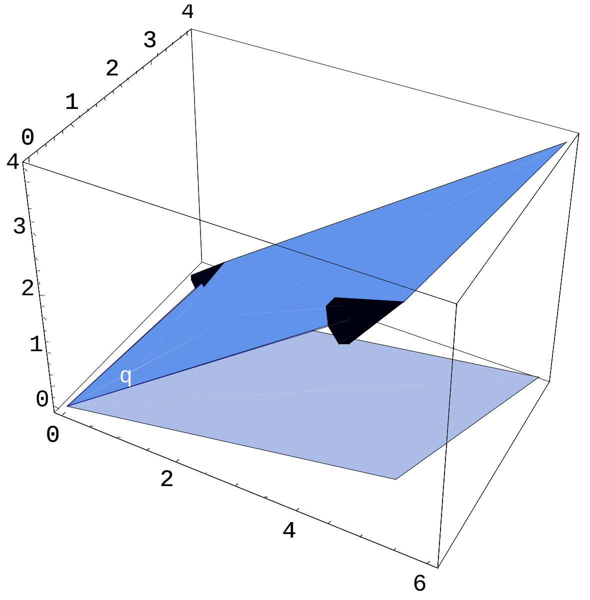



Figure 5

Let's consider two non collinear and non null vectors A and B. These two vectors create a parallelogram which surface equals:

$s = \|A\|\|B\||\sin\theta|$  $\theta$ being the angle between the 2 vectors.

But $\|A \wedge B\| = \|A\|\|B\||\sin\theta|$

Let's consider the parameterized surface:

X(t, u)=(x(t, u), y(t, u), z(t, u))  (t, u)$\in$ A $\subseteq$ $R^2$.

If A and B are the tangents $\frac{\partial X}{\partial t}$ and $\frac{\partial X}{\partial u}$ to this surface at point (t, u), then:

$$\left\|\frac{\partial X}{\partial t} \wedge \frac{\partial X}{\partial u}\right\|$$

is equal to the area of the parallelogram created by $\frac{\partial X}{\partial t}$ and $\frac{\partial X}{\partial u}$;

This parallelogram is part of the tangent plane.

Let's add the following condition:

$$(t_1, u_1) \neq (t_2, u_2) \Rightarrow X(t_1, u_1) \neq X(t_2, u_2)$$

in other terms, the surface is injective or " surface with two faces".

So, if the components x(t, u), y(t, u), z(t, u), (t, u) $\in$ A are of class $C^1$, we define the area of the surface S to be:

$$Area(S) = \iint_S d\sigma = \iint_A \left\| \frac{\partial X}{\partial t} \wedge \frac{\partial X}{\partial u} \right\| \, dtdu$$

Symbolically we can write:

$$d\sigma = \left\| \frac{\partial X}{\partial t} \wedge \frac{\partial X}{\partial u} \right\| dtdu$$

**EXAMPLE**

In the space $(\theta, \varphi, \rho)$ the sphere is reduced to the plane $\rho = a$ that is a 2 sided surface.

Let's evaluate the area of the sphere. We know that:

$$\left\| \frac{\partial X}{\partial \varphi} \wedge \frac{\partial X}{\partial \theta} \right\| = a^2 \sin\varphi \Rightarrow S = \iint_S d\sigma = \iint_R a^2 \sin\varphi d\varphi d\theta = a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi = 4\pi a^2$$

$$R = \{0 \le \varphi \le \pi;\ 0 \le \theta \le 2\pi\}$$

## 2.2 Surface, Cartesian coordinates

Let the surface given in Cartesian coordinates by z= f(x, y). A parameterized representation of this surface is given by:

X(x, y)=(x, y, f(x, y) )

so

$$\frac{\partial X}{\partial x} = (1, 0, f_x^{'}) \text{ et } \frac{\partial X}{\partial y} = (0, 1, f_y^{'}) \Rightarrow \frac{\partial X}{\partial x} \wedge \frac{\partial X}{\partial y} = (-f_x^{'}, -f_y^{'}, 1)$$

$$\Rightarrow \boxed{\left\| \frac{\partial X}{\partial x} \wedge \frac{\partial X}{\partial y} \right\| = \sqrt{1 + f_x^{'2} + f_y^{'2}}}$$

$$\iint_S d\sigma = \iint_A \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dxdy$$

**EXAMPLE 1**

Evaluate the area of the portion of the paraboloid defined by:

$$z = x^2 + y^2, \quad 0 \le z \le 2$$

Using the precedent formula, we get:

$$S=\iint_D \sqrt{1+4x^2+4y^2}\,dxdy \quad with\ D=\{(x,y)/x^2+y^2\le 2\}$$

Using polar coordinates, we get:

$$S=\int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r\sqrt{1+4r^2}\,dr=\frac{13}{3}\pi$$

**EXAMPLE 2**

Evaluate the area of a sphere having the following equation:

$x^2+y^2+z^2=a^2$

In the $(x,y,z)$ space, the sphere is not a two sided surface. It has four, 2 faces for $z=+\sqrt{a^2-x^2-y^2}$ and 2 faces for $z=-\sqrt{a^2-x^2-y^2}$ .

We shall compute the area of the hemisphere $z=\sqrt{a^2-x^2-y^2}$ and multiply the result by 2.

$$z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}};\ z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}};$$

$$\sqrt{1+z'^2_x+z'^2_y}=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

$$S_1=\iint_S d\sigma=\iint_R \sqrt{1+z'^2_x+z'^2_y}\,dxdy=\iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$$

$$=\int_0^{2\pi} d\theta\int_0^a \frac{ar}{\sqrt{a^2-r^2}}dr=2\pi a^2$$

$$S=2S_1=4\pi a^2$$

**IMPORTANT REMARK**

Let S be a two sided surface and $D_{xy}$ its projection on the plane xOy. Let's suppose that the equation of S is given by:

z= f(x, y)

where f is a continuous and bijective function. Let n be the normal to the surface at a given point:

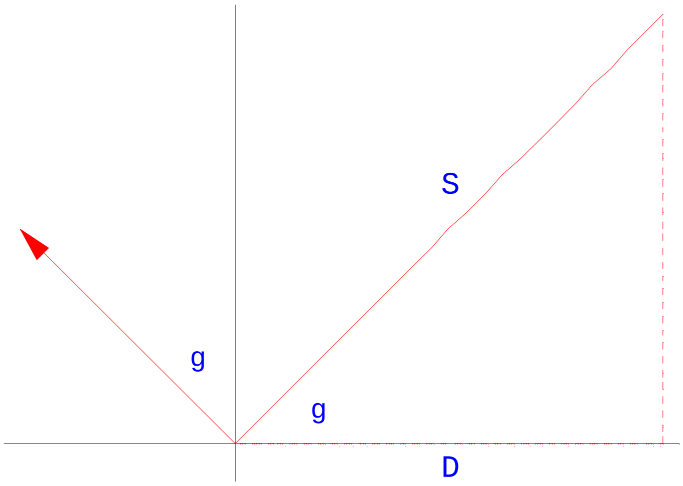



Figure 6

$$\cos\gamma = \left|\frac{D}{S}\right| \Rightarrow S = \left|\frac{D}{\cos\gamma}\right|$$

P= (x, y, f(x, y))

Let's consider at this point an elementary surface $\Delta S$ . We have:

$$\Delta S = \frac{1}{|\cos\gamma|}\Delta D = \frac{1}{|\cos\gamma|}dxdy$$

with $(\cos\alpha, \cos\beta, \cos\gamma)$ coefficients of the normal vector n:

$n = (\cos\alpha, \cos\beta, \cos\gamma)$ => $n.k = (\cos\alpha, \cos\beta, \cos\gamma).(0,0,1) = \cos\gamma$

γ being the angle between the normal vector and the z axis, we can write:

$$\frac{1}{|\cos\gamma|} = \frac{1}{|n.K|} = \frac{1}{\left|\frac{(-f_x', -f_y', 1)}{\sqrt{1 + f_x'^2 + f_y'^2}} \cdot (0,0,1)\right|} = \sqrt{1 + f_x'^2 + f_y'^2}$$

So we can conclude:

$$S = \iint_{D_{xy}} \frac{1}{|\cos\gamma|}dxdy = \iint_{D_{xy}} \sqrt{1 + f_x'^2 + f_y'^2}\,dxdy$$

Thus, if we project the domain over the planes yOz or xOz, we can easily prove that:

$$S = \iint_{D_{yz}} \frac{1}{|\cos\alpha|}dydz = \iint_{D_{yz}} \sqrt{1 + x_y'^2 + x_z'^2}\,dydz$$

$$S = \iint_{D_{xz}} \frac{1}{|\cos\beta|} dxdz = \iint_{D_{xz}} \sqrt{1 + y_x'^2 + y_z'^2}\, dxdz$$

### 2.3 Area of a revolution surface

Let a plane curve z= z(x), going from point $A = (x_o, z_o)$ to point $B = (x_1, z_1)$.
When this curve turns around the z axis, it creates the surface S.

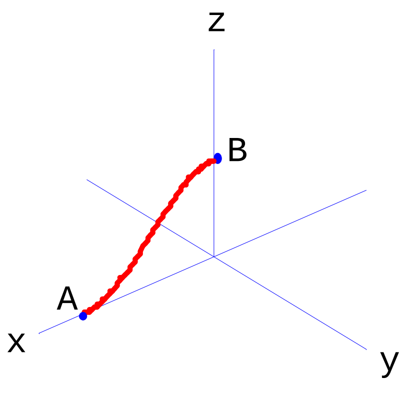



Figure 7

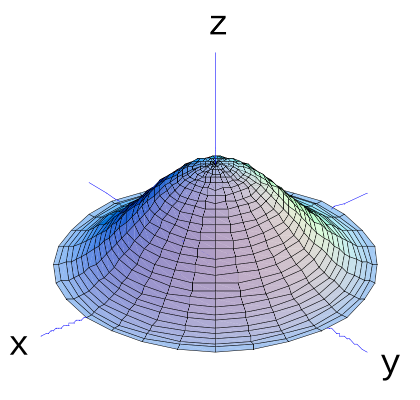



Figure 8

Every point of this curve travels over a circle which perimeter is p(x)=2πx.

The area of this surface is given by:

$$S = \int_A^B 2\pi\ x\ ds$$

ds being the elementary length of the curve z(x). So

$$ds = \sqrt{1 + z_x'^2}\, dx$$

Finally

$$S = \int_{x_0}^{x_1} 2\pi\ x \sqrt{1 + z'^2_x}\, dx$$

## 3. Surface integral

Let S be a parameterized surface given by $X(t,u),\ (t,u) \in A$. Let f be a function defined on S. We define the surface integral of f over S to be:

$$\iint_S f d\sigma = \iint_A f(X(t,u)) \left\| \frac{\partial\, \mathrm{X}}{\partial\, \mathrm{t}} \wedge \frac{\partial\, \mathrm{X}}{\partial\, \mathrm{u}} \right\| dtdu$$

When f=1, the surface integral is the area of the surface S.

**EXAMPLE**

Let S be the surface defined by $z = x^2 + y$ with {0 ≤ x ≤ 1, -1 ≤ y ≤1}. Evaluate the surface integral of the function f(x, y, z) =x.

$$\iint_S x d\sigma = \iint_D x\sqrt{1+z_x'^2 + z_y'^2}\,dxdy = \int_0^1 x\sqrt{2+4x^2}\,dx\int_{-1}^1 dy =$$

$$\frac{1}{3}\sqrt{2}(1+3x^2)^{3/2}\Big]_0^1 = -\frac{\sqrt{2}}{3}+\sqrt{6}$$

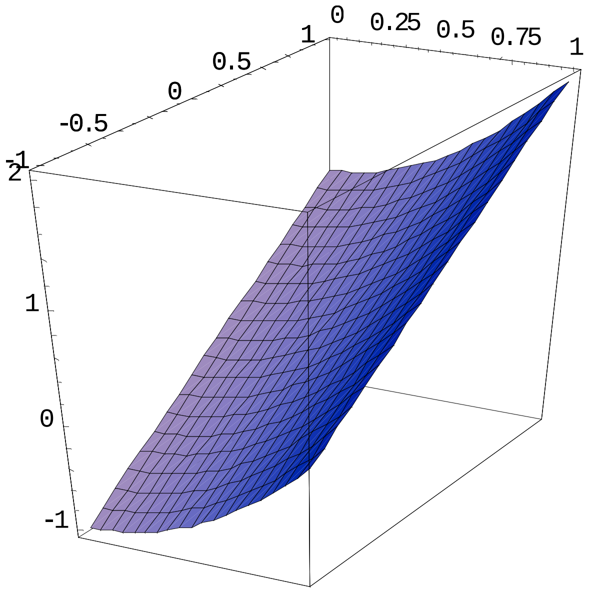



Figure 9

# 4. Flux of a vector field through a surface.

## 4.1 Definition

Let S be a surface, represented parametrically by X(t, u). Let's suppose that S is contained in an open set U of $R^3$. Let F be a vector field of $R^3$ defined in U.

Let n be a normal vector to the surface, directed toward the exterior. The scalar product F.n is the normal component of the field F.

We shall call flux of the field F through the surface S, the quantity

$$flux = \Phi = \iint_S F.n\; d\sigma$$

This flux is therefore equal to:

$$\Phi = \iint_S F.n\; d\sigma = \iint_A F(X(t,u)).\cdot n\left\|\frac{\partial X}{\partial t}\wedge\frac{\partial X}{\partial u}\right\| dtdu$$

with $(t,u)\in A$

but

$$n=\frac{\frac{\partial X}{\partial t}\wedge\frac{\partial X}{\partial u}}{\left\|\frac{\partial X}{\partial t}\wedge\frac{\partial X}{\partial u}\right\|}$$

so

$$\Phi=\iint_S F.n\ ds=\iint_U F(X(t,u))\cdot\frac{\partial X}{\partial u}\wedge\frac{\partial X}{\partial t}dtdu$$

$$\Phi=\iint_S F.n\ ds=\iint_U F\cdot N\, dtdu$$

It's $\frac{\partial X}{\partial t}\wedge\frac{\partial X}{\partial u}$ or $\frac{\partial X}{\partial \mathrm{u}}\wedge\frac{\partial X}{\partial t}$ depending on the orientation of the normal vector.

**REMARK**

It's not always possible to direct the surface in such a way to obtain an interior and an exterior. In this course we shall be limited to cases where geometrically orientation is possible.

**EXAMPLE 1**

Evaluate the flux of the field of vectors: F(x, y, z) = (x, y, 0)

Through the hemisphere $z=\sqrt{a^2-x^2-y^2}$

$$\Phi=\iint_S F.nd\sigma=\iint_D F.Nd\varphi d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/2}a^3\sin^3\varphi d\varphi=2\pi a^3\int_0^{\pi/2}\sin\varphi(1-\cos^2\varphi)d\varphi=\frac{4}{3}\pi a^3$$

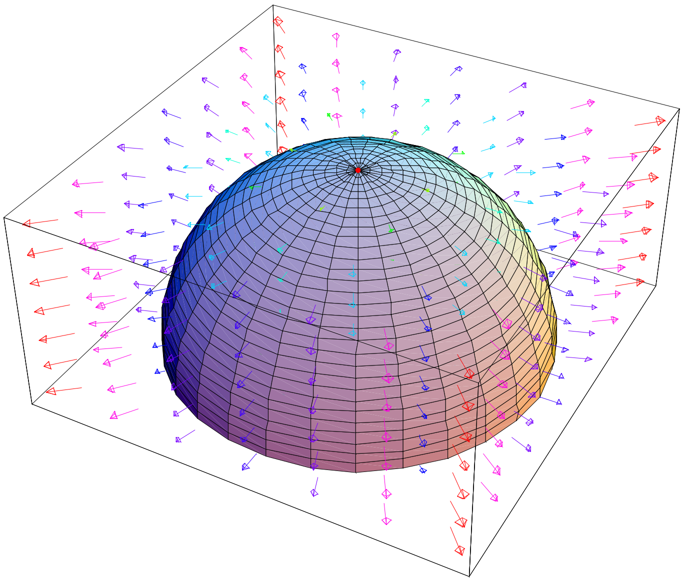


Figure 10

**EXEMPLE 2**

Evaluate the flux of the field of vectors: $F(x, y, z) = (y, -x, z^2)$ through the paraboloid $z = x^2 + y^2; \ 0 \le z \le 1$

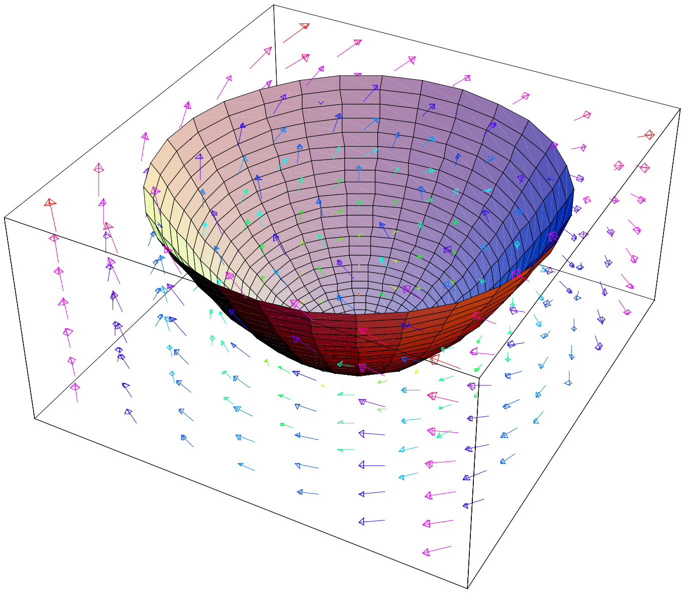


Figure 11

Let's parameterize this paraboloid:

$$X(x, y) = (x, y, x^2 + y^2); \quad (x, y) \in D = \{(x, y) / x^2 + y^2 \le 1\}$$

$$\frac{\partial X}{\partial x} = (1, 0, 2x); \quad \frac{\partial X}{\partial y} = (0, 1, 2y); \Rightarrow \frac{\partial X}{\partial x} \wedge \frac{\partial X}{\partial y} = (-2x, -2y, 1) = \left(-f_x', -f_y', 1\right)$$

This vector is directed toward the interior of the paraboloid, we shall take as normal vector:

$$\frac{\partial X}{\partial y} \wedge \frac{\partial X}{\partial x} = (2x, 2y, -1)$$

So the flux equals:

$$\Phi = \iint_S F \cdot n d\sigma = \iint_D F \cdot N dxdy = -\iint_D (x^2 + y^2)^2 dxdy$$

$$= -\int_0^{2\pi} d\theta \int_0^1 r^5 dr = -\frac{\pi}{3}$$

# Part 2

**LEMMA 1**

We shall call divergence of a vector field F($f_1$, $f_2$, $f_3$) the scalar:

$$divF = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

**LEMMA 2**

We shall call rotational of a vector field U($P, Q, R$) the scalar:

$$rotU = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

## 5. Divergence theorem, Ostrogradski theorem or Green's theorem in space

### 5.1 General case

*Let U be an open set whose boundary consists of a finite number of surfaces. Let F be a vector field on an open set containing U and S. Let n be the unit outward normal vector to S. Then the flux of the field F through the surface S is given by:*

$$\Phi = \iint_S F.n \, d\sigma = \iiint_U divF \; dV$$

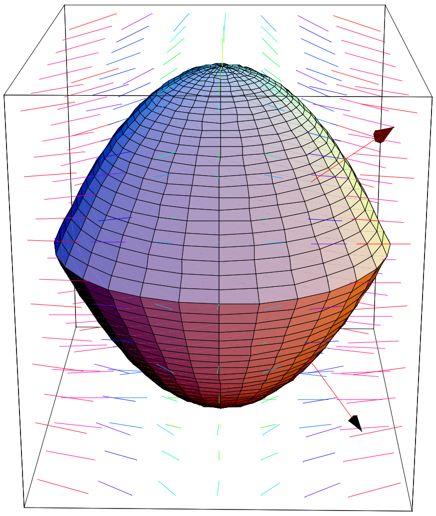


Figure 12

**PROOF**

Let S be a closed surface such that every parallel line to one of the main axis cuts S in two points at the maximum. Let's consider the vector field:

$F = (f_1, f_2, f_3)$

Let's consider that the lower and the upper parts equations are:

$z_1 = g_1(x, y);\ \ z_2 = g_2(x, y)$ and D, the projection of the surface on the plane xOy.

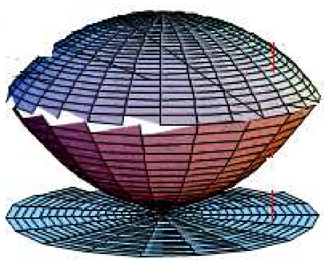


Figure 13

Thus we have:

$$\iiint_U \frac{\partial f_3}{\partial z} dxdydz = \iint_D dxdy \int_{z_1}^{z_2} \frac{\partial f_3}{\partial z} dz$$

$$= \iint_D [f_3(x, y, g_2(x, y)) - f_3(x, y, g_1(x, y))] dxdy$$

For the upper part $S_2$ the normal $n_2$ to $S_2$ makes a sharp angle $\gamma_2$ with k.

$dxdy = \cos\gamma_2 ds_2 \Rightarrow \; dxdy = k.n_2 ds_2$

For the lower part $S_1$:

$dxdy = -\cos\gamma_1 ds_1 \Rightarrow \; dxdy = -k.n_1 ds_1$

since the normal $n_1$ to $S_1$ makes an obtuse $\gamma_1$ with k.

So:
$$\iiint_U \frac{\partial f_3}{\partial z} dxdydz = \iint_{S_2} f_3(x,y,z)k.n_2 ds_2 + \iint_{S_1} f_3(x,y,z)k.n_1 ds_1$$
$$= \iint_S f_3(x,y,z)k.nd\sigma$$

$$\iiint_U \frac{\partial f_3}{\partial z} dxdydz = \iint_S f_3(x,y,z)k.nd\sigma \qquad (1)$$

In the same manner, by projecting S on the other main planes we get:

$$\iiint_U \frac{\partial f_2}{\partial y} dxdydz = \iint_S f_2(x,y,z)\, j.nd\sigma \qquad (2)$$

$$\iiint_U \frac{\partial f_1}{\partial x} dxdydz = \iint_S f_1(x,y,z)i.nd\sigma \qquad (3)$$

By adding (1), (2) and (3), we get:

$$\iiint_U divF \; dV = \iint_S [\, f_1 i \; + f_2 j + f_3 k).nd\sigma = \iint_S F.n \; d\sigma$$

We shall note that the flux can be written:

$$\iint_S F.n \; d\sigma = \iint_S [\, f_1 i \; + f_2 j + f_3 k).nd\sigma$$
$$= \iint_S f_1 i.nd\sigma \; + f_2 j.nd\sigma + f_3 k.nd\sigma$$
$$= \iint_S f_1 \cos\alpha d\sigma \; + f_2 \cos\beta d\sigma + f_3 \cos\gamma d\sigma$$
$$= \iint_S f_1 dydz \; + f_2 dxdz + f_3 dxdy$$

**EXAMPLE 1**

Evaluate the flux of the following field: $F(x,y,z) = \left(x^2, y^2, z^2\right)$ exiting through a unit cube (sol. 3)

**EXAMPLE 2**

Evaluate the flux of the following field:

F(x, y, z) = (x, y, z) exiting through a unit sphere centered at the origin (sol 4π)

## 5.2 General statement

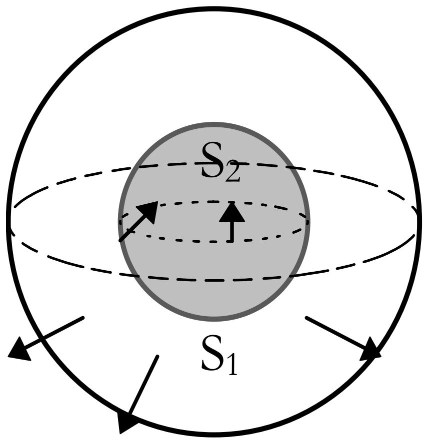
*Let U be a region of the 3-space R3 which the boundary is the union of a finite number of surfaces in such a way that U is in the interior (in the opposite direction of each normal) of every one of the surfaces.*

*Let F be a vector field defined in an open set U containing S. Let n be the unit normal vector directed toward the exterior of S. So:*

$$\Phi = \iint_S F.n\, d\sigma = \iiint_U divF\, dV$$

**EXAMPLE**

Let U be the region bounded by the concentric spheres $S_1$ and $S_2$, and let F be a vector field such that: div F=0




So

$$\iiint_U divF\, dV = 0 \Rightarrow \iint_{S_1} F.nd\sigma + \iint_{S_2} F.nd\sigma = 0 \Rightarrow \iint_{S_1} F.nd\sigma = -\iint_{S_2} F.nd\sigma$$

If we change the orientation of n in $S_2$ we will get:

$\iint_{S_1} F.nd\sigma = \iint_{S_2} F.nd\sigma$ which leads to the following corollary:

### Corollary

*Let S1 and S2 be closed surfaces such that S1 is contained in the interior of S2 and let U be the region between them. If F is a vector field such that: divF=0, then the integral of F over S1 is equal to the integral of F over S2.*

$$\iint_S rotF.n\, d\sigma = \int_C F\, dC$$

## 6. Stokes theorem

*Let S be a 2 sided superficial domain, bounded by a closed curve C.*

*Let's direct C in such a way that S becomes at its left.*

*Let F be a vector field defined on a open set containing S and its boundary.*

*Then:*

$$\iint_S rotF.n\, d\sigma = \int_C F\, dC$$

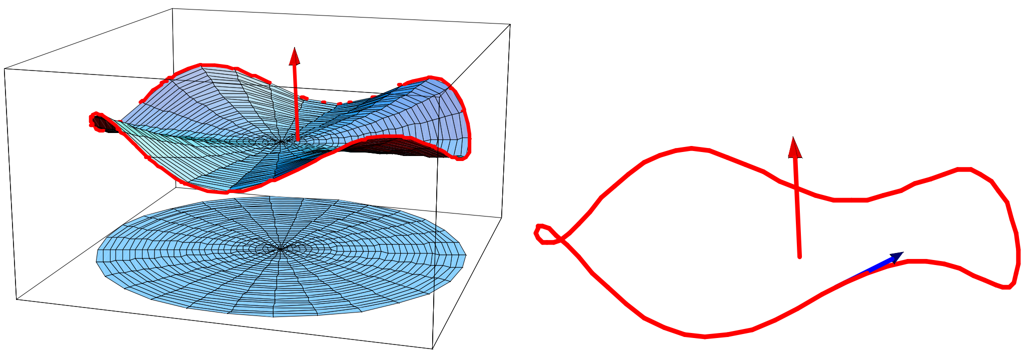


FIGURE 14

**PROOF**

Let S a 2 sided surface given by: z = f(x, y)

Let's assume that S is contained inside the domain U and that C is its boundary.

**Lemma**

The normal to this surface is given by the gradient of "z - f(x, y)=0", then:

$$n = \frac{(-f_x^{'}, -f_y^{'}, 1)}{\sqrt{1 + f_x^{'2} + f_y^{'2}}} \Rightarrow n.i = \frac{-f_x^{'}}{\sqrt{1 + f_x^{'2} + f_y^{'2}}} \text{ and } n.j = \frac{-f_y^{'}}{\sqrt{1 + f_x^{'2} + f_y^{'2}}}$$

Since

$$n.k = \frac{1}{\sqrt{1 + f_x^{'2} + f_y^{'2}}}$$

$$\Rightarrow \; n.i = -f_x^{'} n.k; \text{ and } \quad n.j = -f_y^{'} n.k$$

On the other hand, we know that:

$$\iint_S d\sigma = \iint_D \frac{1}{n.k} dxdy \text{ in other words } \iint_D dxdy = \iint_S n.k d\sigma$$

Thus, let's consider a vector field of class $C^1$ defined on U.

$$F(x,y,z) = \left(X(x,y,z), Y(x,y,z), Z(x,y,z)\right)$$

We need to prove that:

$$\iint_S rotF.n \, d\sigma = \int_C F \, dC$$

Which means:

$$\iint_S \left[\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) i.n + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) j.n + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) k.n\right] d\sigma = \int_C Xdx + Ydy + Zdz$$

Let D be the projection of S on the plane xOy and L the boundary of D.

**Evaluation of** $\int_C Xdx$

$$\int_C X(x,y,z)dx = \int_L X(x,y,f(x,y))dx$$

Let P =X and Q =0, using Green's theorem:

$$\int_C Xdx = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_D \left(-\frac{\partial P}{\partial y}\right) dxdy$$

But $P(x,y) = X\left(x,y,f(x,y)\right) \Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial X}{\partial y} + \dfrac{\partial X}{\partial z}\dfrac{\partial z}{\partial y}$

Then $\displaystyle \int_C Xdx = -\iint_D \left(\frac{\partial X}{\partial y} + \frac{\partial X}{\partial z}\frac{\partial f}{\partial y}\right) dxdy$ ;

$$\iint_D dxdy = \iint_S n.kd\sigma \Rightarrow \quad \int_C Xdx = -\iint_S \left( \frac{\partial X}{\partial y} + \frac{\partial X}{\partial z}\frac{\partial f}{\partial y} \right) n.kd\sigma$$

$$-f_y' n.k = n.j \Rightarrow -\iint_S \left( \frac{\partial X}{\partial y} + \frac{\partial X}{\partial z}\frac{\partial f}{\partial y} \right) n.kd\sigma = -\iint_S \frac{\partial X}{\partial y} n.kd\sigma + \iint_S \frac{\partial X}{\partial z} n.jd\sigma$$

$$\Rightarrow \int_C Xdx = -\iint_S \frac{\partial X}{\partial y} n.kd\sigma + \iint_S \frac{\partial X}{\partial z} n.jd\sigma$$

**Evaluation of** $\int_C Ydy$

$$\int_C Y(x,y,z)dx = \int_L Y(x,y,f(x,y))dx$$

Let Q=Y and P(x, y)=0, using Green's theorem:

$$\int_C Ydy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \iint_D \left( \frac{\partial Q}{\partial x} \right) dxdy$$

But

$$Q(x,y) = Y(x,y,f(x,y)) \quad \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial z}\frac{\partial z}{\partial x}$$

Then

$$\int_C Ydy = -\iint_D \left( \frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial z}\frac{\partial f}{\partial x} \right) dxdy$$

$$\iint_D dxdy = \iint_S n.kd\sigma \Rightarrow$$

$$\int_C Ydy = -\iint_S \left( \frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial z}\frac{\partial f}{\partial x} \right) n.kd\sigma$$

$$-f_x' n.k = n.i \Rightarrow$$

$$-\iint_S \left( \frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial z}\frac{\partial f}{\partial x} \right) n.kd\sigma = -\iint_S \frac{\partial Y}{\partial x} n.kd\sigma + \iint_S \frac{\partial Y}{\partial z} n.id\sigma$$

$$\Rightarrow \int_C Ydy = -\iint_S \frac{\partial Y}{\partial x} n.kd\sigma + \iint_S \frac{\partial Y}{\partial z} n.id\sigma$$

**Evaluation of** $\int_C Zdz$

$$\int_C Zdz = \int_L Z(x,y,f(x,y))dz = \int_L Z(x,y,f(x,y))(f_x^{'}dx + f_y^{'}dy)$$

Let $\begin{matrix} P(x,y) = Z(x,y,f(x,y))\, f_x^{'}(x,y) \\ Q(x,y) = Z(x,y,f(x,y))\, f_y^{'}(x,y) \end{matrix}$, using Green's theorem:

$$\int_C Zdz = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$= \iint_D \left[ \left( \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial z} f_x^{'} \right) f_y^{'} + Zf_{xy}^{"} - \left( \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} f_y^{'} \right) f_x^{'} - Zf_{xy}^{"} \right] dxdy$$

$$= \iint_D \left( \frac{\partial Z}{\partial x} f_y^{'} - \frac{\partial Z}{\partial y} f_x^{'} \right) dxdy$$

$$\int_C Zdz = \iint_D \left( -\frac{\partial Z}{\partial x} n.j - \frac{\partial Z}{\partial y} n.i \right) d\sigma$$

**By adding the precedent integrals:** $\int_C Xdx + Ydy + Zdz$

$$\int_C F = \iint_S \left[ \left( \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right) n.i + \left( \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \right) n.j + \left( \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) n.k \right] d\sigma$$

$$= \iint_S rotF.nd\sigma$$

what needed to be demonstrated

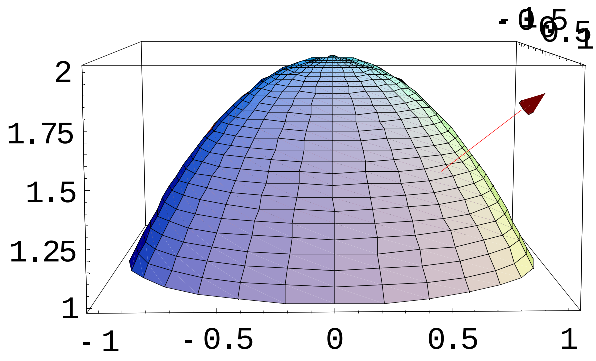
**EXAMPLE**

Verify the Stokes theorem for the following field:

F(x, y, z) = (z - y, x + z, -x - y)

On the domain: $z = 4 - x^2 - y^2; \quad z = 0$

(sol. 8π)




**REMARK**

The Stokes theorem can be used for a surface containing many holes (like the gruyere cheese). The surface integral of rotF.n, is therefore equal to the curve integral along all the boundaries of S.

# Part 3

## 7. Mass, Center of inertia, Moment of inertia

### 7.1 Mass of a twisted plate

We shall call twisted plate every pair $(S,\rho)$ where S is a surface of $R^3$ and $\rho : S \to R_+$ a continuous application called superficial density of the plate.

The mass of the twisted plate $(S,\rho)$ of $R^3$ is the real number **m** defined by:

$$m = \iint_S \rho(M)\, d\sigma$$

where M is a point of S and dS an elementary surface.

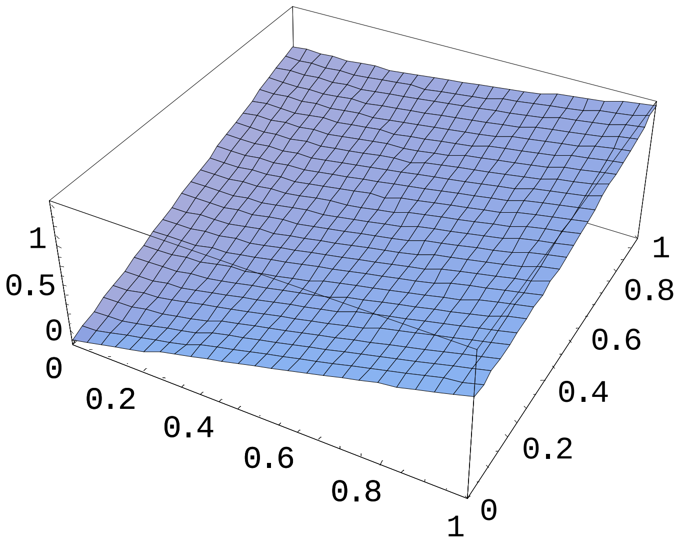
#### 7.1.1 Example

Evaluate the mass of the twisted plate $(S,\rho)$ where S is a piece of the cone defined by:

$$z = \sqrt{x^2 + y^2}; 0 \le x \le 1; 0 \le y \le 1 \text{ et } \rho(x,y,z) = x^2$$

$$m = \iint_D x^2 \sqrt{1 + z_x'^2 + z_y'^2}\, dxdy = \iint_D x^2 \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dxdy = \iint_D x^2 \sqrt{2}\, dxdy$$

$$m = \sqrt{2}\int_0^1 x^2 dx \int_0^1 dy = \frac{\sqrt{2}}{3}$$




## 7.2 Centre of inertia of a twisted plate

The center of inertia of a twisted plate $(S,\rho)$ is the point G of $R^3$ defined by:

$$\begin{cases} x_G = \dfrac{1}{m}\iint\limits_D x\rho(x,y,z)\,d\sigma \\ y_G = \dfrac{1}{m}\iint\limits_D y\rho(x,y,z)\,d\sigma \\ z_G = \dfrac{1}{m}\iint\limits_D z\rho(x,y,z)\,d\sigma \end{cases}$$

For an homogenous plate, we use the term center of gravity instead of center of inertia.

### 7.2.1 Example

Find the center of gravity of the hemisphere:

$z = \sqrt{a^2 - x^2 - y^2}$

having a constant density of 1. We can easily verify that $x_G = y_G = 0$.

$$z_G = \frac{1}{m}\iint_S z\rho d\sigma = \frac{1}{m}\iint_S z\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$$

$$= \frac{1}{m}\iint_S \sqrt{a^2-x^2-y^2}\,\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$$

$$= \frac{a}{m}\iint\limits_D dxdy = \frac{a}{m}\pi a^2 = \frac{a}{2\pi a^2}\pi a^2 = \frac{a}{2}$$

### 7.2.2 Example

Find the center of gravity of the homogenous plate $(S,1)$ where S is the surface defined by: $x = ue^v\cos v; y = ue^v\sin v; z = e^v; (u,v)\in[0,1]\times[0,1]$

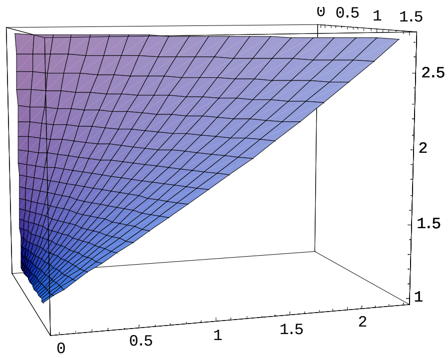



Figure 15

We have:

$$\frac{\partial X}{\partial u}=\begin{pmatrix} e^{v}\cos v \\ e^{v}\sin v \\ 0 \end{pmatrix} \; ; \; \frac{\partial X}{\partial v}=\begin{pmatrix} ue^{v}\left(\cos v-\sin v\right) \\ ue^{v}\left(\sin v+\cos v\right) \\ e^{v} \end{pmatrix}$$

$$\frac{\partial X}{\partial u}\wedge\frac{\partial X}{\partial v}=e^{2v}\begin{pmatrix} \sin v \\ -\cos v \\ u \end{pmatrix} \Rightarrow \left\|\frac{\partial X}{\partial u}\wedge\frac{\partial X}{\partial v}\right\|=e^{2v}\sqrt{1+u^{2}}$$

The mass m is given by:

$$m=\iint_{S}\left\|\frac{\partial X}{\partial u}\wedge\frac{\partial X}{\partial v}\right\|dudv=\left(\int_{0}^{1}\sqrt{1+u^{2}}\,du\right)\left(\int_{0}^{1}e^{2v}dv\right)$$

$$=\frac{1}{4}\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)\left(e^{2}-1\right)\cong 3.667$$

## 7.3 Moment of inertia of a twisted plate:

Let H be a point, a line or a plane of $R^3$; for every point M of $R^3$, we denote $d\left(M,H\right)$ the distance between M and H.

The moment of inertia of the plate $\left(S,\rho\right)$ with respect to H is the real $I_H$ defined by:

$$I_{H}=\iint_{S}\rho\left(M\right)\left(d\left(M,H\right)\right)^{2}d\sigma$$

where $M\left(x,y,z\right)$ travels along S.

### 7.3.1 Example

Evaluate the moment of inertia with respect to the z axis of the homogenous plate $(S,1)$ where S is the surface defined by: $S=\{z=xshy;\quad 0\le x\le 1;\quad 0\le x\le 1\}$

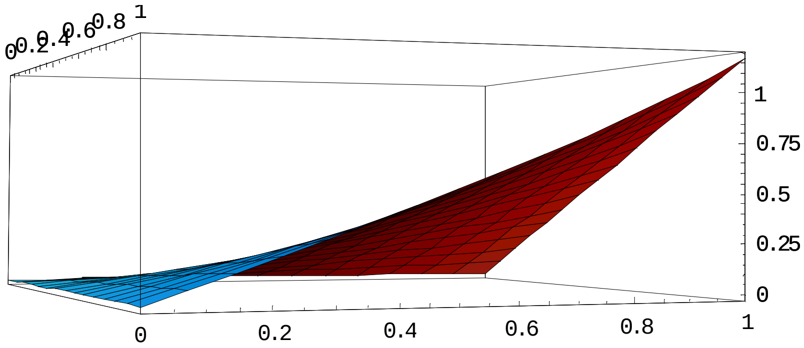



Figure 16

$$\sqrt{1+z_x'^2+z_y'^2}=\sqrt{1+sh^2y+x^2sh^2y}=\sqrt{1+x^2}chy$$

$$I_{z'z}=\iint_S \rho\left(x^2+y^2\right)\sqrt{1+x^2}chydxdy=\left(\int_0^1 x^2\sqrt{1+x^2}dx\right)\left(\int_0^1 chydy\right)$$

$$I_{z'z}=1.59$$