Procédés divers d'intégration

Rappel de cours

1. Intégration des fractions rationnelles

$$\frac{A}{x-a}\ ;\ \frac{A_n}{(x-a)^n}+\frac{A_{n-1}}{(x-a)^{n-1}}+\dots+\frac{A_1}{(x-a)};$$

$$\frac{Ax+B}{x^2+px+q};\ x^2+px+q=\left(x+\frac{p}{2}\right)^2+\frac{1}{4}\left(4q-p^2\right)\ \text{on pose}\ u=x+\frac{p}{2}$$

$$\frac{A_n x+B_n}{\left(x^2+px+q\right)^n}+\frac{A_{n-1}x+B_{n-1}}{\left(x^2+px+q\right)^{n-1}}+\dots+\frac{A_1 x+B_1}{x^2+px+q}$$

2. Intégration par changement de variables: rationalisation

Différentielle contenant seulement des puissances fractionnaires de x $\Rightarrow$ $x=t^n$

Différentielle contenant seulement des puissances fractionnaires de a+bx $\Rightarrow$ $a+bx=t^n$

3. Différentielle binôme

Différentielle binôme de la forme $x^m\left(a+bx^n\right)^{\frac{r}{s}}dx$ dans laquelle m,n,r,s sont des entiers et où n est positif.

1er Cas : $\frac{m+1}{n}=entier \Rightarrow$ on pose $a+bx^n=t^s$

2ème Cas : $\frac{m+1}{n}+\frac{r}{s}=entier\ ou\ 0 \Rightarrow$ on pose $a+bx^n=t^s x^n$

4. Transformation des différentielles trigonométriques

$$Tan\left(\frac{u}{2}\right)=t$$

$$\sin(u)=\frac{2t}{1+t^2};\ \cos(u)=\frac{1-t^2}{1+t^2};\ \mathrm{Tan}(u)=\frac{2t}{1-t^2};\ du=\frac{2dt}{1+t^2}$$

5. Intégration des expressions contenant $\sqrt{a^2-u^2}$ ou $\sqrt{u^2\pm a^2}$

Quand $\sqrt{a^2-u^2}$ se présente, on pose $u=a\sin(z)$

Quand $\sqrt{u^2+a^2}$ se présente, on pose $u=aTan(z)$

Quand $\sqrt{u^2-a^2}$ se présente, on pose $u=\frac{a}{\cos^2(z)}$

CORRIGE

Exercice 1

$$\int \frac{8}{x^3-4x}dx$$

$$\frac{8}{x^3-4x}=\frac{1}{x-2}-\frac{2}{x}+\frac{1}{x+2}\Rightarrow \int \frac{8}{x^3-4x}dx=Log\left(x^2-4\right)-2Log\left(x\right)$$

$$\int \frac{5x^2-9}{x^3-9x}dx$$

$$\frac{5x^2-9}{x^3-9x}=\frac{2}{x-3}+\frac{1}{x}+\frac{2}{x+3}\Rightarrow \int \frac{5x^2-9}{x^3-9x}dx=Log\left(x\right)+2Log\left(x^2-9\right)$$

$$\int \frac{x+2}{x^4+2x^3+x^2}dx$$

$$\frac{x+2}{x^4+2x^3+x^2}=\frac{2}{x^2}-\frac{3}{x}+\frac{1}{(x+1)^2}+\frac{3}{x+1}$$

$$\int \frac{x+2}{x^4+2x^3+x^2}dx=3Log\left(x+1\right)-3Log\left(x\right)-\frac{1}{x+1}-\frac{2}{x}$$

$$\int \frac{6x^2+3x+4}{x^3+2x}dx$$

$$\frac{6x^2+3x+4}{x^3+2x}=\frac{2}{x}+\frac{4x+3}{x^2+2}$$

$$\Rightarrow \int \frac{6x^2+3x+4}{x^3+2x}dx=\frac{3}{\sqrt{2}}ArcTan\left(\frac{x}{\sqrt{2}}\right)+2Log\left(x\right)+2Log\left(x^2+2\right)$$

Exercice 2

$$\int \frac{4x^3+3x^2+18x+12}{(x^2+4)^2}dx$$

$$\frac{4x^3+3x^2+18x+12}{(x^2+4)^2}=\frac{2(x+2)}{\left(x^2+4\right)^2}+\frac{2(2x+1)}{x^2+4}$$

$$\int \frac{4x^3+3x^2+18x+12}{(x^2+4)^2}dx=\frac{x-2}{2\left(x^2+4\right)}+\frac{5}{4}ArcTan\left(\frac{x}{2}\right)+2Log\left(x^2+4\right)$$

$$\int_0^{\pi/4} e^x(\sqrt{3}\cos 2x-\sin 2x)dx$$

$$\int e^x(\sqrt{3}\cos 2x-\sin 2x)dx=\frac{\sqrt{3}}{2}e^xCos(x)+\frac{2}{5}e^xCos(2x)+\frac{\sqrt{3}}{2}Sin(x)-\frac{1}{5}e^xSin(2x)$$

$$\int_0^{\pi/4} e^x(\sqrt{3}\cos 2x-\sin 2x)dx=\frac{1}{10}\left(5\sqrt{6}-2\right)e^{\frac{\pi}{4}}-\frac{1}{10}\left(4+5\sqrt{3}\right)$$

$$\int_0^1\left(x^2e^x+\frac{x^2}{2}-\ln(1+x)\right)dx$$

$$\int(x^2e^x+\frac{x^2}{2}-\ln(1+x))dx=x+\frac{x^3}{6}+e^x\left(2-2x+x^2\right)-Log(1+x)-xLog(1+x)$$

$$\int_0^1(x^2e^x+\frac{x^2}{2}-\ln(1+x))dx=\frac{1}{6}\left(6e-5-12Log(2)\right)$$

Exercice 3

$$\int \frac{5x^2+12x+9}{x^3+3x^2+3x}dx$$

$$\frac{5x^2+12x+9}{x^3+3x^2+3x}=\frac{3}{x}+\frac{3+2x}{3+3x+x^2}$$

$$\int \frac{5x^2+12x+9}{x^3+3x^2+3x}dx=3Log(x)+Log\left(3+3x+x^2\right)$$

$$\int \frac{5x+9}{(x-9)x^{3/2}}dx$$

Posons $x = t^2$

$$\int \frac{5x+9}{(x-9)x^{3/2}}dx = \int\left(\frac{2}{t-3}-\frac{2}{t^2}-\frac{2}{t+3}\right)dt = 2\left(\frac{1}{t}+Log\,(t-3)-Log\,(t+3)\right)$$

$$\int \frac{5x+9}{(x-9)x^{3/2}}dx = \frac{2}{\sqrt{x}}+2Log\left(\sqrt{x}-3\right)-2Log\left(\sqrt{x}+3\right)$$

$$\int \frac{\sqrt{x}}{x^3+2x^2-3x}dx$$

Posons $x = t^2$

$$\int \frac{\sqrt{x}}{x^3+2x^2-3x}dx = \int \frac{t}{t^6+2t^4-3t^2} = \int\left(\frac{1}{4(t-1)}-\frac{1}{4(t+1)}-\frac{1}{2(t^2+3)}\right)$$

$$= 2\left(\frac{1}{4\sqrt{3}}ArcTan\left(\frac{t}{\sqrt{3}}\right)+\frac{1}{8}Log(t-1)-\frac{1}{8}Log(t+1)\right)$$

$$= -\frac{1}{2\sqrt{3}}ArcTan\left(\frac{\sqrt{x}}{3}\right)+\frac{1}{4}Log\left(\sqrt{x}-1\right)-\frac{1}{4}Log\left(\sqrt{x}+1\right)$$

$$\int \frac{dx}{x-x^{4/3}}$$

Posons $x = t^2$

$$\int \frac{dx}{x-x^{4/3}} = \int \frac{1}{t^3-t^4}dt = \int\left(\frac{2}{t^2}+\frac{2}{t}-\frac{2}{t-1}\right)dt$$

$$= 2\left(-\frac{1}{t}-Log(t-1)+Log(t)\right)$$

$$= -3Log\left(x^{1/3}-1\right)+3Log\left(x^{1/3}\right)$$

Exercice 4

$$\int_0^3 \frac{dx}{(x+2)\sqrt{x+1}}$$

Posons $x = t^2$

$$\int \frac{dx}{(x+2)\sqrt{x+1}} = \int \frac{2}{t^2+1}dt = 2ArcTan(t) \Rightarrow \int_0^3 \frac{dx}{(x+2)\sqrt{x+1}} = 2\left[ArcTan(2) - \frac{\pi}{4}\right]$$

$$\int_0^{1/2} \frac{dx}{\left(9+\sqrt[3]{2x}\right)\sqrt{2x}}$$

Posons $x = \frac{t^3}{2}$;

$$\int \frac{dx}{\left(9+\sqrt[3]{2x}\right)\sqrt{2x}} = \frac{3}{2}\left(2\sqrt{t} - 6ArcTan\frac{\sqrt{t}}{3}\right)$$

$$\int_0^{1/2} \frac{dx}{\left(9+\sqrt[3]{2x}\right)\sqrt{2x}} = \frac{3}{2}\left(\sqrt{2} - 6ArcTan\frac{1}{3\sqrt{2}}\right)$$

$$\int_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}dx$$

Posons $x = t^3 + 2$

$$\int \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}dx = \int\left(\frac{27}{t^2+3} + 3t^2 - 9\right)dt = 3\left(3\sqrt{3}ArcTan\left(\frac{t}{\sqrt{3}}\right) + \frac{t^3}{3} - 3t\right)$$

$$\int_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3}dx = 3\left(\sqrt{3}\pi + \frac{1}{3}\left(8 - \frac{3\sqrt{3}\pi}{2}\right)\right)$$

$$\int x^5\sqrt{1+x^3dx}$$

$$x^m\left(a+bx^n\right)^{\frac{r}{s}}dx$$

Exercice 5

$$\int \frac{x^5}{(2+3x^3)^{3/2}}dx$$

$$x^m\left(a+bx^n\right)^{\frac{r}{s}}dx$$

Ici $m=5;\ n=3;\ r=-3;\ s=2 \Rightarrow \frac{m+1}{n} + \frac{r}{s} = \frac{5+1}{3} = 2 = entier$

$\Rightarrow$ On pose $a+bx^n = t^s \Rightarrow 2+3x^3 = t^2 \Rightarrow t=\sqrt{2+3x^3}$

$$\int \frac{x^5}{(2+3x^3)^{3/2}}dx = \int \frac{(t^2-2)^{5/3}}{3\times 3^{2/3}t^3}\frac{2t}{3\times 3^{1/3}(t^2-2)^{2/3}}dt = \int \frac{2t^3-4t}{27t^3}dt = \frac{2}{27}t - \frac{4}{27}\frac{1}{t}+c$$

$$\Rightarrow \int \frac{x^5}{(2+3x^3)^{3/2}}dx = \frac{4}{27\sqrt{2+3x^3}} + \frac{2}{27}\sqrt{2+3x^3}+c$$

$$\int \frac{dx}{x^3(1+x^3)^{1/3}}$$

Ici $m=-3;\ n=3;\ r=-1;\ s=3 \Rightarrow \frac{m+1}{n}+\frac{r}{s} = \frac{-3+1}{3}-\frac{1}{3} = -1$

$\Rightarrow$ On pose $a+bx^n = t^s x^n \Rightarrow 1+x^3 = t^3x^3 \Rightarrow x = \sqrt[3]{\frac{1}{t^3-1}} \Rightarrow t=\sqrt[3]{\frac{1+x^3}{x^3}}$

$$\Rightarrow dx = \frac{-t^2}{(t^3-1)^{4/3}}dt \Rightarrow \Rightarrow \frac{dx}{x^3(1+x^3)^{1/3}} = \frac{(t^3-1)\times(t^3-1)^{\frac{1}{3}}}{t}\frac{-t^2}{(t^3-1)^{4/3}}dt = -tdt$$

$$\int \frac{dx}{x^3(1+x^3)^{1/3}} = \int -tdt = \frac{t^2}{2}+c = -\frac{1}{2}\left(\frac{1+x^3}{x^3}\right)^{2/3}+c = \frac{(1+x^3)^{2/3}}{2x^2}+c$$

$$\int \frac{dx}{1+\sin x+\cos x}$$

Posons $t = Tan\left(\frac{x}{2}\right)$

$$\sin(x) = \frac{2t}{1+t^2};\quad \cos(x) = \frac{1-t^2}{1+t^2};\ Tan(x) = \frac{2t}{1-t^2};\quad du = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2}{1+t^2}}{1+\frac{2t+1-t^2}{1+t^2}}dt = \int \frac{dt}{1+t} = Log(1+t)+c$$

$$Log(1+t)+c = Log\left(\cos\left(x/2\right)+\sin\left(x/2\right)\right) - Log\left(\cos\left(x/2\right)\right)+c$$

$$\int \frac{dx}{\sin x + tgx}$$

Posons $t = Tan\left(\frac{x}{2}\right)$

$$\sin(x) = \frac{2t}{1+t^2};\quad \cos(x) = \frac{1-t^2}{1+t^2};\; Tan(x) = \frac{2t}{1-t^2};\quad dx = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{\sin x + tgx} = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2} + \frac{2t}{1-t^2}} dt = \int \frac{1-t^2}{2t} dt = \frac{Log[t]}{2} - \frac{t^2}{4} + c$$

$$\int \frac{dx}{\sin x + tgx} = \frac{Log\left(Tan\left(x/2\right)\right)}{2} - \frac{\left(Tan\left(x/2\right)\right)^2}{4} + c$$

Exercice 6

$$\int \frac{dx}{5 + 4\cos x}$$

Posons $t = Tan\left(\frac{x}{2}\right) \Rightarrow$

$$\sin(x) = \frac{2t}{1+t^2};\quad \cos(x) = \frac{1-t^2}{1+t^2};\; Tan(x) = \frac{2t}{1-t^2};\quad du = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{5 + 4\cos x} = \int \frac{\frac{2}{1+t^2}}{5 + 4\frac{1-t^2}{1+t^2}} dt = \int \frac{2}{9+t^2} dt = \frac{2}{9} ArcTan\left(\frac{1}{3} Tan\left(\frac{x}{2}\right)\right) + c$$

$$\int \frac{dx}{1 + \sin x - \cos x}$$

Posons $t = Tan\left(\frac{x}{2}\right)$

$$\int \frac{dx}{1 + \sin x - \cos x} = \int \frac{\frac{2}{1+t^2}}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} dt = \int \frac{1}{t^2 + t} dt = \int \left(\frac{1}{t} - \frac{1}{1+t}\right) dt$$

$$\int \frac{dx}{1 + \sin x - \cos x} = Log(t) - Log(1+t) + c = Log\left(Tan\left(x/2\right)\right) - Log\left(1 + Tan\left(x/2\right)\right) + c$$

$$\int_0^{\pi} \frac{dx}{5-3\cos x}$$

Posons $t = Tan\left(\frac{x}{2}\right)$

$$\int \frac{dx}{5-3\cos x} = \int \frac{\frac{2}{1+t^2}}{5-3\frac{1-t^2}{1+t^2}} dt = \int \frac{2}{5+5t^2-3+3t^2} dt = \int \frac{1}{1+4t^2} dt$$

$$\int_0^{\pi} \frac{dx}{5-3\cos x} = \frac{1}{2} ArcTan[2t]\Big]_0^{\pi} = \frac{1}{2} ArcTan[2\pi]\Big]$$

$$\int_0^{\pi/2} \frac{dx}{2+\cos x}$$

$$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} dt = \int \frac{2}{3+t^2} dt = \frac{2}{\sqrt{3}} ArcTan\left(\frac{t}{\sqrt{3}}\right) + c$$

$$\int_0^{\pi/2} \frac{dx}{2+\cos x} = \frac{2}{\sqrt{3}} ArcTan\left(\frac{Tan\left(\frac{\pi}{4}\right)}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} ArcTan\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

Exercice 7

$$\int_0^{\pi/2} \frac{dx}{2+\sin x}$$

Posons $t = Tan\left(\frac{x}{2}\right) \Rightarrow$

$$\sin(x) = \frac{2t}{1+t^2};\quad \cos(x) = \frac{1-t^2}{1+t^2};\ Tan(x) = \frac{2t}{1-t^2};\quad du = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{2+\sin x} = \int \frac{\frac{2}{1+t^2}}{2+\frac{2t}{1+t^2}} dt = \int \frac{dt}{1+t+t^2} = \int \frac{dt}{\frac{3}{4}+\left(\frac{1}{2}+t\right)^2}$$

$$\int \frac{dt}{\frac{3}{4}+\left(\frac{1}{2}+t\right)^2} = \frac{2}{\sqrt{3}} \int \frac{du}{1+u^2} = \frac{2}{\sqrt{3}} ArcTan\left(\frac{1+2Tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)$$

$$\int_0^{\pi/2} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$$

$$\int_0^{\pi/6} \frac{1+tgx}{1+\sin 2x} dx$$

Posons $t = Tan(x) \Rightarrow \sin(2x) = \frac{2t}{1+t^2}$; $\quad dx = \frac{dt}{1+t^2}$

$$\int_0^{\pi/6} \frac{1+tgx}{1+\sin 2x} dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t}{1+\frac{2t}{1+t^2}} \frac{dt}{1+t^2} = \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t}{1+t^2+2t} dt = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(1+t)} dt$$

$$\int_0^{\pi/6} \frac{1+tgx}{1+\sin 2x} dx = Log(1+t)\Big]_0^{1/\sqrt{3}} = Log\left(1+\frac{1}{\sqrt{3}}\right)$$

$$\int_0^1 \frac{x-2}{(2x-3)^2} dx;$$

$$\frac{x-2}{(2x-3)^2} = \frac{1}{2(2x-3)} - \frac{1}{2(2x-3)^2}$$

$$\int_0^1 \frac{x-2}{(2x-3)^2} dx = \int_0^1 \left(\frac{1}{2(2x-3)} - \frac{1}{2(2x-3)^2}\right) dx = \frac{1}{4} Log|2x-3| + \frac{1}{4(2x-3)}\Big]_0^1$$

$$\int_0^1 \frac{x-2}{(2x-3)^2} dx = -\frac{Log(3)}{4} - \frac{1}{6}$$

$$\int_0^1 \frac{e^x - 1}{e^x + 1} dx$$

$$\int_0^1 \frac{e^x - 1}{e^x + 1} dx = \int_0^1 \left(1 - \frac{2}{e^x + 1}\right) dx = \left[-x + 2Log\left(1 + e^x\right)\right]_0^1 = -1 - \ln 4 + 2Log\left(1 + e\right)$$